Quadratic Nonlinear Control of a Self-excited Oscillator
Ayman A. El-Badawy and Tarek N. Nasr El-Deen
*Journal of Vibration and Control* 2007; 13; 403
DOI: 10.1177/1077546307076283

The online version of this article can be found at:
http://jvc.sagepub.com/cgi/content/abstract/13/4/403

Published by:
© SAGE Publications
http://www.sagepublications.com

Additional services and information for *Journal of Vibration and Control* can be found at:

Email Alerts: http://jvc.sagepub.com/cgi/alerts
Subscriptions: http://jvc.sagepub.com/subscriptions
Reprints: http://www.sagepub.com/journalsReprints.nav
Permissions: http://www.sagepub.com/journalsPermissions.nav
Quadratic Nonlinear Control of a Self-excited Oscillator

AYMAN A. EL-BADAWY
Department of Engineering Design and Production Technology, The German University in Cairo, Cairo, Egypt (ayman.elbadawy@guc.edu.eg)

TAREK N. NASR EL-DEEN
Mechanical Engineering Department, Al-Azhar University, Cairo, Egypt

(Received 30 September 2005; accepted 16 November 2006)

Abstract: The objective of this work is to demonstrate the feasibility of the use of the nonlinear saturation-based control concept to suppress self-excited vibrations by means of active nonlinear feedback control. The authors use the van der Pol oscillator as the working model for a self-excited system. A saturation phenomenon is induced by tuning the frequency of the under-damped second-order absorber to one-half that of the primary system. Although the authors conclude that we can achieve some level of performance with this control technique, we question its robustness due to the rich dynamics introduced by the controller.

Key words: Saturation, self-excited system, quadratic absorber.

1. INTRODUCTION

The force acting on a vibrating system is usually external to the system and independent of its motion. However, there are systems for which the exciting force is a function of some part of the state of the system, such as its displacement, velocity, and acceleration. Such systems are called self-excited oscillators, because the force that sustains the motion is created or controlled by the motion itself. In many cases, a self-excited system possesses negative linear damping and positive nonlinear damping. A positive damping force is directed opposite to the velocity and a negative damping force is directed along the velocity.

Self-excited oscillations resulting from a form of negative damping occur in many physical systems. The instability of rotating shafts, the flutter of wings and turbine blades, the flow-induced vibration of pipes, the aerodynamically induced motion of bridges, the galloping of cables, and the chattering of machine tools are all typical examples of self-excited vibrations. A number of textbooks (see, e.g., Nayfeh and Mook, 1979; Nayfeh, 1981), deal with self-excited systems and the methods used to investigate them. Thus, it would be of much interest to develop new ways of reducing the persistent oscillations of such systems.

Parametric vibrations are induced by a varying system parameter (e.g. stiffness). On the other hand, autoparametric vibration (self parametric) is characterized by an internal coupling involving at least two modes. From a mathematical point of view, this coupling is induced by nonlinear terms in the equations of motion of the combined system. Physically
speaking, an autoparametric system consists of two parts: A main system and a secondary system. When the main system is externally excited, the secondary system is parametrically excited as a result of the variation of its stiffness by the response of the main system. In other words, a two-mode interaction occurs when the main system exhibits a forced response which, in turn, drives the secondary system into parametric resonance. In this case, energy is transferred from one part of the combined system to the other. This energy transfer depends on the type of nonlinearities and the damping forces, and can be partial or complete, depending on the system parameters.

In the case of quadratic nonlinearities and free vibrations, the energy transfer is complete when the natural frequency of the main system is exactly twice the natural frequency of the secondary system; this case is referred to as a two-to-one internal resonance, (Nayfeh and Mook, 1979; Nayfeh, 2000). In the case of forced vibrations, the response of the main system can be suppressed. When the main system is excited at a frequency near its natural frequency (i.e., primary resonance), the response of the main system will have the same frequency as the excitation, and the response amplitude will increase linearly with the amplitude of excitation. However, above a critical value, the response of the main system saturates and all of the additional energy from the exciter is channeled to the secondary system. This is referred to as the saturation phenomenon and has been developed as a nonlinear control strategy by Oueini et al. (Oueini et al., 1998, 1999; Oueini and Nayfeh, 2000). A similar control strategy of self-excited vibrations was implemented by Verhulst (2005), who quenched undesirable vibrations by energy absorption through embedding the oscillator in an autoparametric system by coupling to a damped oscillator.

Hall et al. (2001) applied the nonlinear saturation phenomenon to suppress the flutter of a wing in numerical simulations. They illustrated the concept by means of an example with a rather flexible, large-span wing of the type found on such vehicles as high endurance aircraft and sailplanes. They concluded that to control the amplitude response, they need to actively adapt the controller frequency because of the nonlinear nature of the systems.

In this article, the authors will evaluate the method of saturation-based control developed by Oueini and Nayfeh (2000) for suppression of the large-amplitude vibrations of a van der Pol self-excited system. We will evaluate the feasibility of the control law for various parameter combinations.

2. MATHEMATICAL MODEL

Considering a van der Pol oscillator, (Nayfeh, 1981), coupled with a saturation-based controller (Oueini et al., 1998), we obtain the following system:

\[
\ddot{x} + \omega_1^2 x = \mu_1 \dot{x} - \mu_3 x^2 \dot{x} + \delta_1 y^2 \tag{1}
\]

\[
\ddot{y} + \omega_2^2 y = -\mu_2 \dot{y} + \delta_2 xy. \tag{2}
\]

Here, \( x \) and \( y \) denote the generalized coordinates of the oscillator and absorber respectively, and \( \omega_1 \) and \( \omega_2 \) are their linear undamped natural frequencies. The overdot represents differentiation with respect to time \( t \). For the energy dissipation oscillator, we incorporate the negative linear damping term \( \mu_1 \dot{x} \), and the positive nonlinear damping term \( \mu_3 x^2 \dot{x} \). For the absorber, we incorporate a positive linear damping term \( \mu_2 \dot{y} \). Finally, the two quadratic
coupling terms $\delta_1 y^2$ and $\delta_2 x y$ are necessary for the energy exchange between the oscillator and absorber to take place. The authors will study these equations using both numerical and analytical techniques. To begin the analytic studies, we must first scale the governing equations. The scaled equations are

$$\ddot{x} + \omega_1^2 x = \varepsilon \mu_1 \dot{x} - \varepsilon \mu_3 x^2 \dot{x} + \varepsilon \delta_1 y^2$$

$$\ddot{y} + \omega_2^2 y = -\varepsilon \mu_1 \dot{y} + \varepsilon \delta_2 x y$$

where $\varepsilon$ is a small bookkeeping parameter. To express the nearness of $\omega_1$ to $2\omega_2$, we write

$$\omega_1 = 2\omega_2 + \varepsilon \sigma .$$

We attack the problem using the method of multiple scales (Nayfeh, 1981), and assume that $x$ and $y$ can be approximated by the following expansions:

$$x \approx x_0 (T_0, T_1) + \varepsilon x_1 (T_0, T_1)$$

$$y \approx y_0 (T_0, T_1) + \varepsilon y_1 (T_0, T_1).$$

Where $T_0 = t$ and $T_1 = \varepsilon t$. Substituting the above expansions into the scaled equations and equating coefficients of like powers of $\varepsilon$, we obtain the following systems of differential equations, which are solved in cascade:

$O(1)$:

$$D_0^2 x_0 + \omega_1^2 x_0 = 0$$

$$D_0^2 y_0 + \omega_2^2 y_0 = 0$$

$O(\varepsilon)$:

$$D_0^2 x_1 + \omega_1^2 x_1 = -2D_0 D_1 x_0 + \mu_1 D_0 x_0 - \mu_3 x_0^2 D_0 x_0 + \delta_1 y_0^2$$

$$D_0^2 y_1 + \omega_2^2 y_1 = -2D_0 D_1 y_0 - \mu_1 D_0 y_0 + \delta_2 x_0 y_0.$$
\begin{align*}
D_0^2 x_1 + \omega_1^2 x_1 &= \left( -2i\omega_1 D_1 A + i\mu_1 \omega_1 A - i\mu_3 \omega_1 A^2 \hat{A} + \delta_1 B^2 e^{-i\sigma T_1} \right) e^{i\omega_1 T_0} \\
&+ \text{NST} + \text{cc} \quad (12) \\
D_0^2 y_1 + \omega_2^2 y_1 &= \left( -2i\omega_2 D_1 B - i\mu_1 \omega_2 B + \delta_2 \tilde{A} \tilde{B} e^{i\sigma T_1} \right) e^{i\omega_2 T_0} + \text{NST} + \text{cc} \quad (13)
\end{align*}

where NST stands for the nonsecular terms and cc stands for the complex conjugate of the preceding terms. In order to obtain a uniform expansion up to first order, the terms in brackets must vanish. This condition yields the modulation equations for the complex-valued amplitudes \(A\) and \(B\):

\begin{align*}
2i\omega_1 A' &= i\mu_1 \omega_1 A - i\mu_3 \omega_1 A^2 \hat{A} + \delta_1 B^2 e^{-i\sigma T_1} \quad (14) \\
2i\omega_2 B' &= -i\mu_1 \omega_2 B + \delta_2 \tilde{A} \tilde{B} e^{i\sigma T_1}. \quad (15)
\end{align*}

Where primes are derivatives with respect to the slow time scale \(T_1\). Expressing (14) and (15) in terms of polar coordinates gives

\begin{align*}
A &= \frac{1}{2} a e^{i\alpha} \\ B &= \frac{1}{2} b e^{i\beta} \quad (16) \quad (17)
\end{align*}

where \(a, b, \alpha, \text{ and } \hat{\alpha}\) are unknown real-valued functions of \(T_1\). Substituting these equations for \(A\) and \(B\) into (16) and (17), we obtain

\begin{align*}
i\omega_1 \left(i\alpha a' + \alpha' \right) e^{i\alpha} &= \frac{1}{2} i\mu_1 \omega_1 a e^{i\alpha} - \frac{1}{8} i\mu_3 \omega_1 a^3 e^{i\alpha} + \frac{1}{4} \delta_1 b^2 e^{2i\beta} e^{-i\sigma T_1} \quad (18) \\
i\omega_2 \left( i\beta b' + b' \right) e^{i\beta} &= -\frac{1}{2} i\mu_1 \omega_2 b e^{i\beta} + \frac{1}{4} \delta_2 a b e^{i(\alpha - \beta) e^{i\sigma T_1}.} \quad (19)
\end{align*}

Separating the real and imaginary parts of (18) and (19), we obtain the following system of real-valued modulation equations:

\begin{align*}
a' &= \frac{1}{2} \mu_1 a - \frac{1}{8} \mu_3 a^3 - \frac{\delta_1 b^2}{4\omega_1} \sin (\sigma T_1 + \alpha - 2\beta) \quad (20) \\
b' &= -\frac{1}{2} \mu_1 b + \frac{\delta_2 ab}{4\omega_2} \sin (\sigma T_1 + \alpha - 2\beta) \quad (21) \\
a \alpha' &= -\frac{\delta_1 b^2}{4\omega_1} \cos (\sigma T_1 + \alpha - 2\beta) \quad (22) \\
b \beta' &= -\frac{\delta_2 ab}{4\omega_2} \cos (\sigma T_1 + \alpha - 2\beta). \quad (23)
\end{align*}
These modulation equations represent a non-autonomous system; to obtain an equivalent autonomous system, we define

\[ \gamma = \sigma T_1 + \alpha - 2\beta. \]  

(24)

Thus, the modulation equations are reduced from a system of four first-order differential equations into the following set of three equations:

\[
\begin{align*}
da' &= \frac{1}{2} \mu_1 a - \frac{1}{8} \mu_3 a^3 - \frac{\delta_1 b^2}{4\omega_1} \sin \gamma \\
b' &= -\frac{1}{2} \mu_c b + \frac{\delta_2 ab}{4\omega_2} \sin \gamma \\
ab \gamma' &= ab\sigma + \left( \frac{\delta_2 a^2}{2\omega_2} - \frac{\delta_1 b^2}{4\omega_1} \right) b \cos \gamma.
\end{align*}
\]

(25) \quad (26) \quad (27)

The equilibrium solution (fixed points) is obtained by setting \( a' = b' = \gamma' = 0 \), giving

\[
\begin{align*}
\frac{1}{2} \mu_1 a - \frac{1}{8} \mu_3 a^3 - \frac{\delta_1 b^2}{4\omega_1} \sin \gamma &= 0 \quad (28) \\
-\frac{1}{2} \mu_c b + \frac{\delta_2 ab}{4\omega_2} \sin \gamma &= 0 \quad (29) \\
ab\sigma + \left( \frac{\delta_2 a^2}{2\omega_2} - \frac{\delta_1 b^2}{4\omega_1} \right) b \cos \gamma &= 0. \quad (30)
\end{align*}
\]

From equations (28) to (30), we obtain the following simplified equations for \( a, b \) and \( \gamma \), which are the response amplitudes of the plant and controller, and a measure of the phase between the responses, respectively.

\[
\begin{align*}
\delta_2^2 \mu_c^5 (a^2)^3 &= -4 \left\{ \omega_2^2 \mu_c^2 \mu_3 + 2\delta_2 (\mu_1 - \mu_c) \mu_3 \right\} (a^2)^2 \\
+ 16 &\left( \mu_1 - \mu_c \right) \left[ \delta_2^2 (\mu_1 - \mu_c) + 2\omega_2^2 \mu_c^2 \mu_3 \right] a^2 \\
- 4\omega_2^2 \mu_c^2 &\left[ 16 (\mu_1 - \mu_c)^2 + 64\sigma^2 \right] = 0 \quad (31)
\end{align*}
\]

\[
b^2 = \frac{\omega_1 \delta_2}{\omega_2 \delta_1 \mu_c} \left( \mu_1 - \frac{1}{4} \mu_3 a^2 \right) a^2 \quad (32)
\]

\[
\sin \gamma = \frac{2\mu_c \omega_2}{\delta_2 a}. \quad (33)
\]

Since (31) is a cubic equation in \( a^2 \), we expect up to three real solutions for the plant response, excluding the trivial one. Numerically, we can determine the roots of the cubic equation (the \( a \) values) and then solve for \( b \) and \( \gamma \) from (32) and (33).
We can also try to solve (28) to (30) analytically. Here, to minimize the math, the authors consider only the case in which the controller is perfectly tuned (i.e., \( \sigma = 0 \)). Again, there are two primary cases: The controller is deactivated, and \( b = 0 \), or the controller is activated, and \( b \neq 0 \). When the controller is activated, (29) requires that \( a \neq 0 \) when \( \mu_c \neq 0 \). When \( \sigma = 0 \), the four possible solutions are

(i) \( a = 0 \), \( b = 0 \) \( \gamma \) arbitrary

(ii) \( a = \left( \frac{4\mu_1}{\mu_3} \right)^{\frac{1}{2}} \), \( b = 0 \) \( \gamma \) arbitrary

(iii) \( a = \frac{2\omega_2 \mu_c}{\delta_2} \), \( b = \left[ \frac{\omega_1 \omega_2 \mu_c \left( 4\mu_1 - \mu_3 \alpha^2 \right)}{\delta_1 \delta_2} \right]^{\frac{1}{2}} \), \( \gamma = \frac{\pi}{2} \)

(iv) \( a = \left[ \frac{4\mu_1 - 8\mu_c}{\mu_3} \right]^{\frac{1}{2}} \), \( b = \left[ \frac{2\omega_2 \delta_2}{\omega_2 \delta_2} \right]^{\frac{1}{2}} \alpha \), \( \gamma = \sin^{-1} \left( \frac{2\omega_2 \mu_c}{\delta_2 a} \right) \)

Solution (i) is the trivial solution, solution (ii) corresponds to the uncontrolled response, solution (iii) is the saturated solution, and solution (iv) is another controlled solution. Solution (iii) is identified as the saturated solution because the amplitude of the plant response \( a \) is independent of the original system parameters \( \omega_1, \mu_1, \) and \( \mu_3 \), and so it will not change even if these are varied. The amplitude \( a \) of solution (iii) is specified by the controller damping \( \mu_c \) and the controller feedback \( \delta_2 \), and can, thus, be made as small as desired. The additional energy that would normally be dissipated by the plant response is channeled into the controller through the two-to-one internal resonance (i.e., through the quadratic coupling terms).

Both controlled solutions exist only in certain regions of the parameter space. Solution (iii) exists when \( \mu_3 < \mu_{32} \) (condition 1) where

\[
\mu_{32} = \frac{\mu_1 \delta_2^2}{\omega_2^2 \mu_c^2}
\]

and solution (iv) exists only when \( \mu_1 > 2\mu_c \) (condition 2). The next section considers the influence of both \( \mu_3 \) and \( \mu_c \) on the response behavior of the system.

3. STABILITY ANALYSIS

To determine the stability of the solutions, we introduce a small perturbation to the fixed points:

\[
a = a_0 + \delta a \\
b = b_0 + \delta b \\
\gamma = \gamma_0 + \delta \gamma.
\]
Then, we substitute the above expansions into equations (25) to (27), linearize with respect to the perturbations $\delta a$, $\delta b$, and $\delta \gamma$, and obtain the following equations to describe the evolution of the perturbations:

$$
\delta a' = \left[ \frac{1}{2} \mu_1 - \frac{3}{8} \mu_3 a_0^2 \right] \delta a - \left[ \frac{2 \delta_1}{4 \omega_1} b_0 \sin \gamma_0 \right] \delta b - \left[ \frac{\delta_1}{4 \omega_1} b_0^2 \cos \gamma_0 \right] \delta \gamma \quad (37)
$$

$$
\delta b' = \left[ \frac{\delta_2}{4 \omega_2} b_0 \sin \gamma_0 \right] \delta a - \left[ \frac{1}{2} \mu_1 c - \frac{\delta_2}{4 \omega_2} a_0 \sin \gamma_0 \right] \delta b - \left[ \frac{\delta_2}{4 \omega_2} a_0 b_0 \cos \gamma_0 \right] \delta \gamma \quad (38)
$$

$$
a_0 b_0 (\delta \gamma)' = \left[ \left( \sigma + \frac{4 \delta_2}{4 \omega_2} a_0 \cos \gamma_0 \right) b_0 \right] \delta a + \left[ a_0 \sigma \left( \frac{\delta_2}{2 \omega_2} a_0^2 - \frac{3 \delta_1}{4 \omega_1} b_0^2 \right) \cos \gamma_0 \right] \delta b - \left[ \frac{\delta_2}{2 \omega_2} a_0^2 - \frac{\delta_1}{4 \omega_1} b_0^2 \right] b_0 \sin \gamma_0 \delta \gamma \quad (39)
$$

We determine the stability of the four solutions by substituting each into equations (37) to (39) and calculating whether the perturbations grow or decay over time. For solution (i) we find that

\[
(\delta a)' = \frac{1}{2} \mu_1 \delta a
\]

\[
(\delta b)' = -\frac{1}{2} \mu_1 \delta b
\]

and hence the solution is unstable because a perturbation will grow. Next, we consider solution (ii):

\[
(\delta a)' = -\mu_1 \delta a
\]

\[
(\delta b)' = -\left[ \frac{1}{2} \mu_1 c - \frac{\delta_2}{4 \omega_2} \left( \frac{4 \mu_1}{\mu_3} \right) \frac{1}{2} \sin \gamma_0 \right] \delta b
\]

\[
0 = \frac{2 \delta_2}{\omega_2 \mu_3} \cos \gamma_0 \delta b
\]

but $\delta b \neq 0$, therefore the third equation indicates that $\cos \gamma_0 = 0$ and we have

\[
(\delta a)' = -\mu_1 \delta a
\]
Thus, solution (ii) is stable when $\mu_3 > \mu_{32}$.

The stability of the other solutions was determined numerically by calculating the eigenvalues of equations (37) to (39) (Nayfeh, 1995). This is necessary because some of the equations do not uncouple, and we obtain a cubic polynomial for the eigenvalues that is difficult to solve in closed form. If the real part of every eigenvalue is negative, the corresponding equilibrium solution is asymptotically stable. If the real part of any of the eigenvalues is positive, the corresponding equilibrium solution is unstable.

Figures 1 and 2 present the effect of tuning the absorber’s frequency away from exactly half the oscillator frequency on the amplitude response of both the oscillator and the absorber for two different gains. The dashed lines denote unstable equilibrium solutions and the solid lines denote stable equilibrium solutions. For both the oscillator and absorber responses, there exists one response that corresponds to the trivial solution and another response that is close to the trivial solution. Both of these responses are unstable. The third and fourth amplitude responses are similar but one is the mirror image of the other. By slowly increasing the value of the frequency $\sigma$ from –8 to 8 the authors encounter supercritical pitchfork bifurcation $PF_1$ at about $\sigma = –4$ for the third solution. Then we encounter a reverse pitchfork bifurcation $PF_2$ for the fourth solution. For the cases of $PF_1$ and $PF_2$, we have one branch of stable fixed points on one side of the bifurcation that becomes unstable due to the bifurcation and another created stable branch whose amplitude of response monotonically decreases for the oscillator and increases for the absorber. Thus it is clear that at $\sigma = 0$ there is only one stable solution, while for any other $\sigma$ there are two stable solutions that coexist and can lead to jumps in the amplitude response of both the oscillator and the absorber. It is clear from this conclusion that this is not a robust method of control since many potential disturbances exist that can change the value of $\sigma$, leading to an unpredictable response. Figure 2 shows the same behavior for the amplitude responses of the oscillator and the absorber as the feedback gain increases to a value of five instead of one. The only difference is the response of the absorber, which is reduced by a factor proportional to the gain. Thus no real improvements in the oscillator response occur, and the choice of gain can be decided upon based on the practical implementation of the absorber circuit and the actuator used.

Next, the authors examine the influence of the nonlinear damping $\mu_3$ on the system response. Figures 3 and 4 show the four different solutions that were determined analytically earlier for the case of $\sigma = 0$. Again, solid lines represent stable solutions while dashed lines represent unstable ones. It is clear from the figures that only solution (iv) is stable, and that as the nonlinear damping coefficient increases, the response amplitude decreases for both the system and the absorber. The only difference between Figures 3 and 4 is the value of the linear damping factor of the system $\mu_1$. We can see that the response amplitude of Figure 4 is less than that of Figure 3, which is to be expected because less energy is needed (due to lower value of $\mu_1$) for the flow of energy between the two modes to occur.

Plots of the fixed-point solutions as a function of $\mu_1$ are shown in Figure 5. The shaded area represents the region where the Hopf-bifurcation solutions exist. Reverse pitchfork bifurcations exist at $\mu_1$ of about 4.4, after which only solution (ii) exists, obeying condition (1). Solution (iv) exists only for values of $\mu_1 < 0.6$, thus abiding by condition (2).
Figure 1. Fixed-point solutions and their stability as a function of the frequency detuning $\sigma$ when $\omega_2 = 30$, $\delta_1 = 4\omega_1$, $\delta_2 = 120$, $\mu_1 = 0.02\omega_1$, $\mu_2 = 1$, and $\mu_3 = 0.005$. Solid lines represent stable solutions, dashed lines represent unstable solutions.

Figure 2. Fixed-point solutions and their stability as a function of the frequency detuning $\sigma$ when $\omega_2 = 30$, $\delta_1 = 20\omega_1$, $\delta_2 = 120$, $\mu_1 = 0.02\omega_1$, $\mu_2 = 1$, and $\mu_3 = 0.005$. Solid lines represent stable solutions, dashed lines represent unstable solutions.
Figure 3. Fixed-point solutions and their stability as a function of the nonlinear damping coefficient $\mu_1$ at $\sigma = 0$ when $\omega_2 = 30$, $\dot{\phi}_1 = 240$, $\dot{\phi}_2 = 120$, $\mu_1 = 1.2$, and $\mu_c = 0.005$. Solid lines represent stable solutions, dashed lines represent unstable solutions.

Figure 4. Fixed-point solutions and their stability as a function of the nonlinear damping coefficient $\mu_1$ at $\sigma = 0$ when $\omega_2 = 30$, $\dot{\phi}_1 = 240$, $\dot{\phi}_2 = 120$, $\mu_1 = 0.6$, and $\mu_c = 0.005$. Solid lines represent stable solutions, dashed lines represent unstable solutions.
Figure 5. Fixed-point solutions as a function of the controller damping coefficient $\mu_c$ at $\sigma = 0$ when $\omega_2 = 30$, $\delta_1 = 240$, $\delta_2 = 120$, $\mu_1 = 1.2$, and $\mu_3 = 1$. Solid lines represent stable solutions, dashed lines represent unstable solutions, and the shaded area is the region of Hopf bifurcated solutions.

Figure 6. Fixed-point solutions and their stability as a function of the nonlinear damping coefficient $\mu_c$ at $\sigma = 0$ when $\omega_2 = 30$, $\delta_1 = 240$, $\delta_2 = 120$, $\mu_1 = 1.2$, and $\mu_3 = 2$. Solid lines represent stable solutions, dashed lines represent unstable solutions.
Finally, in Figure 6, and using the results from Figure 5, the controller damping, $\mu_c$, is chosen ($\mu_c = 2$) to insure that the system operates in the region where solution (ii) is unstable, solution (iii) is stable and is the desired saturated solution, and solution (iv) does not exist. The feedback gain ($k_2$) can be chosen to reduce the amplitude of the oscillations to any desired value and is only limited by the available control authority of the actuator. Again, solid lines represent stable solutions while dashed lines represent unstable ones.

4. CONCLUSION

From this analysis of the nonlinear feedback control implementation, we conclude that it is possible to obtain system responses that are independent of the original system parameters when the controller is tuned to exactly half the natural frequency of the plant. This is the saturated solution. Second, the amplitude of the controlled response is never larger than the amplitude of the uncontrolled system, although the improvement is sometimes small. Third, the addition of a saturation-based controller to a self-excited system may add an abundance of interesting dynamics to the system which, brings into question the robustness of that control strategy.

REFERENCES