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ACTIVE CONTROL OF FREE BEAM VIBRATIONS UTILIZING ACTUATOR OPTIMIZATION

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Abstract

For this investigation, a collocated piezoceramic sensor/actuator pair to control the free vibrations of a pinned-pinned Euler-Bernoulli beam is used. The technique of positive position feedback control is employed; the piezo location and size is optimized. The control “goodness” is quantified by the total amount of damping to the different modes of the system.

1. INTRODUCTION AND MODELING

1.1 Piezoceramic Sensor and Actuator Models

We will be using collocated piezoelectric sensors and actuators to control the vibrations of a pinned-pinned beam. A detailed explanation of the use and modeling of piezoelectric sensors and actuators for beams is given by Crawley and Anderson [4].

Adali, et al [5] optimized the size and location of a piezo-actuator to minimize the deflection of a frame structure under bending load. The results are given for deterministic and uncertain loading conditions.

Adachi, et al [6] developed a design method of the active/passive hybrid type of piezoelectric damping system for reducing the dynamic response of a flexible structure due to external dynamic loads. Their design method based on the numerical optimization technique whose function is a control of the active damping.

For this work, we will assume that the sensor voltage is proportional to the curvature of the beam

$$V_s \propto F(\omega''(x,t)) \quad (1)$$

Later we will identify the constant of proportionality which we now label as a_1 . Furthermore, we assume that the actuator can be modeled as a uniform moment distributed over the length of the piezo and proportional to the applied voltage:

$$M(x, t) = a_1 V_a(t) [u(x - x_1) - u(x - x_2)] \quad (2)$$

where a_1 is the constant of proportionality, $V_a(t)$ is the applied voltage, (x_1, x_2) are the beginning and ending positions of the piezoactuator and $u(x)$ is the unit step function.

1.1 Beam Model

The governing equations for an Euler-Bernoulli beam subject to an external moment are:

$$[EI(x) \omega''(x, t)]'' + \mu(x) \ddot{\omega}(x, t) = M''(x, t) \quad (3)$$

where the primes denote differentiation with respect to x , dots denote differentiation with respect to time and $M(x, t)$ is the applied moment due to the piezoactuator as given in the previous section. The boundary conditions are:

$$\omega(0, t) = \omega''(x, t) = 0 \quad \text{and} \quad \omega(L, t) = \omega''(L, t) = 0$$

Next we discretize the system by expanding the deflection $w(x, t)$ in terms of the mass normalized modeshapes of the uniform beam. That is,

$$\omega(x, t) = \sum \phi_j(x) z_j(t) \quad (4)$$

where the modeshapes $\phi_j(x)$ are given by

$$\phi_j(x) = \sqrt{\frac{2}{\mu L_b}} \sin\left(\frac{j\pi x}{L_b}\right)$$

with

$$\int_0^{L_b} \mu(x) \phi_i(x) \phi_j(x) dx = \delta_{ij}$$

$$\int_0^{L_b} EI(x) \phi_i''(x) \phi_j''(x) dx = \omega_j^2 \delta_{ij}$$

where the natural frequencies of vibration ω_j are given by

$$\omega_j = (j\pi)^2 \sqrt{\frac{EI}{\mu L_b^4}}$$

Retaining only a finite number of terms N_m in the expansion we obtain:

$$\ddot{\mathbf{z}}(\mathbf{t}) + \Omega_k \mathbf{z}(\mathbf{t}) = a_1 \mathbf{B} V_a(t) \quad (5)$$

where Ω_k is the diagonal matrix of eigenvalues ω_j^2 and the RHS is the forcing term due to the actuator. The vector \mathbf{B} is given by

$$\mathbf{B} = \phi'_j(x_2) - \phi'_j(x_1) \quad (6)$$

and is a modal participation vector.

2. POSITIVE POSITION FEEDBACK CONTROL

Following the lead of Fanson and Caughey [1] we rewrite our governing equations as

$$\ddot{\mathbf{z}}(\mathbf{t}) + \Omega_k \mathbf{z}(\mathbf{t}) = a_1 C^T \mathbf{G} \mathbf{n}(t) \quad (7)$$

which prepares us to apply positive position feedback control. The new variables are:

$$\begin{aligned} \mathbf{n}(t) &= N_f \times 1 && \text{vector of filter coordinates} \\ G &= N_f \times N_f && \text{diagonal matrix of gains} \\ C &= N_f \times N_m && \text{modal participation matrix} \end{aligned}$$

where N_m is the number of modes and N_f is the number of filters. It is obvious that the true input voltage $V_a(t)$ is related to the filter coordinates by:

$$\mathbf{B} V_a(t) = C^T \mathbf{G} \mathbf{n}(t) \quad (8)$$

If we define the new vector $\hat{\mathbf{B}} = \frac{\mathbf{B}^T}{\mathbf{B} \mathbf{B}^T}$ such that $\hat{\mathbf{B}}^T \mathbf{B} = 1$, a scalar, then the actual necessary control voltage is

$$V_a(t) = \hat{\mathbf{B}} C^T \mathbf{G} \mathbf{n}(t)$$

The filter coordinates $\mathbf{n}(t)$ are governed by:

$$\ddot{\mathbf{n}}(t) + \tilde{\Lambda}_D \dot{\mathbf{n}}(t) + \tilde{\Omega}_k \mathbf{n}(t) = a_2 \tilde{G} C \mathbf{z}(t) \quad (9)$$

We will choose $\tilde{G} = \tilde{\Omega}_k$ as did Fanson and Caughey [1] to reduce the number of unknowns in our problem. The resulting discrete problem is

$$\ddot{\mathbf{z}}(\mathbf{t}) + \Omega_k \mathbf{z}(\mathbf{t}) = a_1 C^T \mathbf{G} \mathbf{n}(t) \quad (10)$$

$$\ddot{\mathbf{n}}(t) + \tilde{\Lambda}_D \dot{\mathbf{n}}(t) + \tilde{\Omega}_k \mathbf{n}(t) = a_2 \tilde{\Omega}_k C \mathbf{z}(t) \quad (11)$$

where only the diagonal matrix G is unknown once we have chosen our filters (i.e., (i.e., $\tilde{\Lambda}_D$ and $\tilde{\Omega}_k$).

Note that we are using collocated piezoelectric devices for both sensing and actuation so

we have

$$V_a \propto a_1 C \quad \text{and} \quad V_s \propto a_2 C$$

that is, both the actuator voltage V_a and the sensor voltage V_s are proportional to the same beam functional which we represent in discrete form as the modal participation matrix C .

We have not identified the modal participation matrix C as of yet. Fanson and Caughey [1] experimentally determined this matrix for their problem. According to them, it was very difficult to experimentally determine \mathbf{B} (i.e., the slopes of the modeshapes) because of the discontinuities at x_1 and x_2 . For their system, these discontinuities caused them to distrust their analytic solutions at these locations as well. In our experience, however, the piezo thickness does not appreciably alter the first or second modeshapes and as we do not have an experimental setup we will rely on the analytic expressions. Part of the functionality of the PPF method, however, is that it can be “calibrated” to the system by using measured frequencies and modal participation matrices.

Because each of our filters will influence the system through the same physical actuator, we postulate that the influence of each filter on each mode is the same, i.e.

$$C = \begin{bmatrix} B^T \\ B^T \\ \vdots \\ B^T \end{bmatrix}$$

or each row of C (corresponding to each filter) will have as its components the components of the modal participation vector \mathbf{B} . Note that this choice for C implies that the actuator voltage V_a from equation (8) becomes:

$$V_a(t) = g_1 n_1(t) + g_2 n_2(t) + \dots + g_{N_m} n_{N_m}(t)$$

3. PROBLEM SOLUTION

Our goal is to use a single piezoelectric sensor/actuator to add damping to several modes of a system. In order to judge the performance, we will plot the real part of the closed-loop system eigenvalues versus actuator position and size just as done by Devasia et al. [2].

To solve the system, we rewrite equations (10) and (11) as follows:

$$\begin{Bmatrix} \dot{\mathbf{z}}(t) \\ \ddot{\mathbf{n}}(t) \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \tilde{\Lambda}_D \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{z}}(t) \\ \dot{\mathbf{n}}(t) \end{Bmatrix} + \begin{bmatrix} \Omega_k & -a_1 C^T G \\ -a_2 \tilde{\Omega}_k C & \tilde{\Omega}_k \end{bmatrix} \begin{Bmatrix} \mathbf{z}(t) \\ \mathbf{n}(t) \end{Bmatrix} = 0 \quad (12)$$

which has the solution

$$\begin{Bmatrix} \mathbf{z}(t) \\ \mathbf{n}(t) \end{Bmatrix} = \mathbf{A} e^{\lambda t} \quad (13)$$

Substituting equation (13) into equation (12) yields the following matrix equation which determines the eigenvalues of the closed-loop system and $N_m + N_f - 1$ of the constants \mathbf{A} .

$$\left\{ \begin{bmatrix} I & 0 \\ 0 & \tilde{I} \end{bmatrix} \lambda^2 + \begin{bmatrix} 0 & 0 \\ 0 & \tilde{\Lambda}_D \end{bmatrix} \lambda + \begin{bmatrix} \Omega_k & -a_1 C^T G \\ -a_2 \tilde{\Omega}_k C & \tilde{\Omega}_k \end{bmatrix} \right\} \mathbf{A} e^{\lambda t} = 0 \quad (14)$$

The eigenvalues are determined by enforcing a nontrivial solution for the vibrations. Once the λ_i have been found, the constants \mathbf{A} can all be found to within a multiplicative constant. We are only interested in the closed-loop eigenvalues, however, so if we define the matrix Γ as follows:

$$\Gamma = \begin{bmatrix} I\lambda^2 + \Omega_k & -a_1 C^T G \\ -a_2 \tilde{\Omega}_k C & \tilde{I}\lambda^2 + \tilde{\Lambda}_D \lambda + \tilde{\Omega}_k \end{bmatrix} \quad (15)$$

where I is an $N_m \times N_m$ identity matrix and \tilde{I} is an $N_f \times N_f$ identity matrix, then the eigenvalues λ_i are the values which make Γ a singular matrix, i.e., $\det \Gamma = 0$

Table 1. Numerical Values.

Beam Properties	Piezo Properties
$E_b = 70$ GPa	$E_p = 63$ GPa
$\mu = 1.25$ kg/m	$d_{31} = 120 \times 10^{-12}$ m/V
$L_b = 0.5$ m	$g_{31} = 10.5 \times 10^{-3}$ Vm/N
$b = 0.05$ m	$C_p = 35$ pC
$t_b = 0.01$ m	$b = 0.05$ m
	$t_p = 0.0002$ m

In the case where we have 3 structural modes and 3 control filter modes we can simply evaluate the determinant of Γ , and then numerically solve for the eigenvalues. For larger systems, we would have to develop some sort of minimization algorithm to search for the λ_i which make Γ singular. Actually, due to the nature of the problem, we can extend this procedure for slightly larger systems as described in Appendix I.

3.1 Stability Analysis

Fanson and Caughey [1] have shown that the system is stable only when

$$S \equiv \Omega_k - C^T G C > 0 \quad (0.1)$$

that is, S is positive definite. The matrix S is a real symmetric matrix, so we can guarantee positive definiteness by either [3]: 1. making all the eigenvalues of S be real and positive, or 2. enforcing the condition that all the principle minor determinants of S be positive, i.e.,

$$\det S_{qr} > 0 \quad \text{for } q = r$$

where S_{qr} is the matrix formed by eliminating the q^{th} column and r^{th} row from S . It is important to note that the stability of the system depends on the actuator size and location directly and therefore must be determined for each actuator combination. Because we will be calculating the system eigenvalues directly we will simply examine the real parts to determine stability in this analysis.

3.1 Numerical Example

For our numerical example, we will use the same physical system used by Devasia et al. [2] as shown in Table 1.

Using these values and the relationships also found in Devasia et al. [2] we calculate the piezo constants of proportionality to be

$$a_1 = 0.0247 \text{ m}^2 / \text{s}^2 \text{ V} \quad \text{and} \quad a_2 = 75714$$

4. RESULTS AND CONCLUSIONS

For this assignment, we will only investigate the effect of piezo size and location for one specific set-up. In particular, we have a 3 mode structural discretization with 3 PPF filters. The filters are exactly tuned to the structural frequencies, i.e.

$$\hat{\omega}_i = \omega_i$$

where ω_i is the frequency of the i^{th} structural mode and $\hat{\omega}_i$ is the corresponding filter frequency. Furthermore, the filter damping was arbitrarily set to

$$2\hat{\xi}_1\hat{\omega}_1 = 0.73, \quad 2\hat{\xi}_2\hat{\omega}_2 = 0.75, \quad 2\hat{\xi}_3\hat{\omega}_3 = 0.81$$

Finally the control gains were chosen, again arbitrarily, to be:

$$g_1 = 0.80, \quad g_2 = 0.60, \quad g_3 = 5.00$$

The results of these controllers on the three structural modes are depicted in Figure 1. Figure (1a) shows the real part of the first structural mode eigenvalue versus the piezo actuator center location. The solid line is for a piezo length of $L_p = 0.05L_b$ with the other lines representing $L_p = 0.25L_b, 0.50L_b,$ and $0.75L_b$. Figures (1b) and (1c) show the same results for structural modes 2 and 3 respectively. The mode 3 plot only shows the lines corresponding to $L_p = 0.05L_b, 0.50L_b,$ and $0.75L_b$ for clarity.

For this particular system, we see that mode 1 is always stable and that the best performance is obtained when the piezo straddles the center of the beam. Interestingly, this model predicts that a small actuator can perform as well as a larger one so long as it is carefully centered.

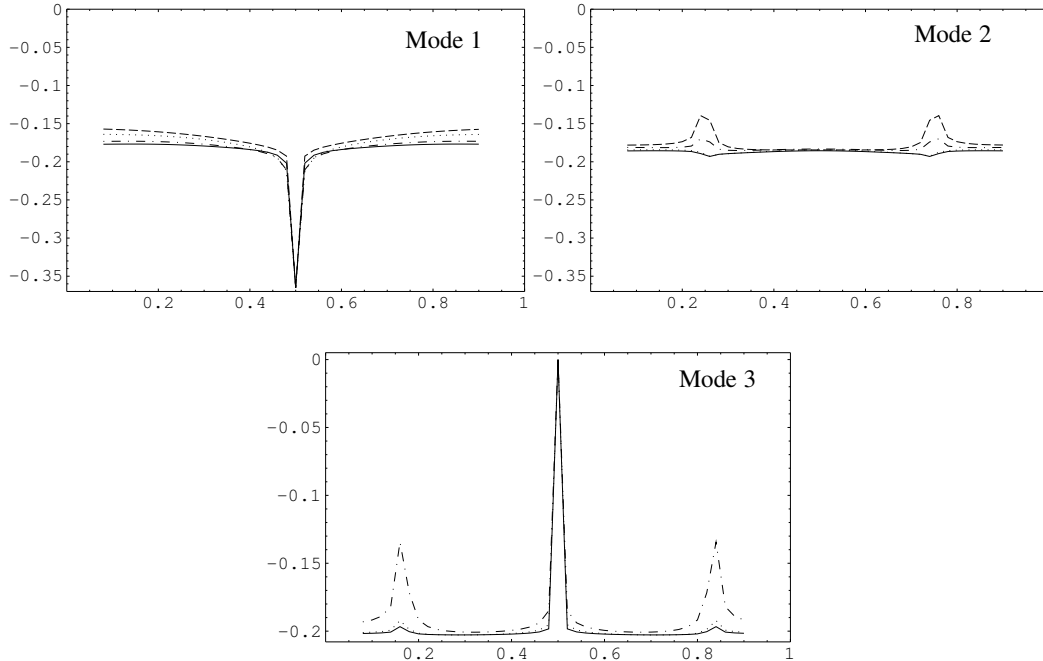


Figure 1. The diagrams above show the real part of the structural eigenvalues as a function of the piezo-actuator center location for several different piezo lengths denoted by the different lines in each graph.

The mode 2 response shows two locations where the actuator can be placed to obtain good performance. These locations correspond to the anti-nodes of the mode. Interestingly, even when the actuator is centered on the beam, and therefore, over a second mode node, the mode remains stable.

The mode 3 response appears to be the most critical. As with the other cases, the smallest actuator appears to work nearly as well as the larger ones, but here we see that centering the actuator on the beam causes the system to lose stability in this mode. There are two other locations corresponding to the two other nodes of the third mode where the actuator will degrade the system performance, but not lose stability.

These results are similar in nature to those presented by Devasia et al. [2], who used only two modes, but are difficult to directly compare as they plotted the minimum eigenvalues for all the modes versus actuator location whereas we show each mode response individually. Their results do indicate, however, that a larger piezo, on the order of 65% to 75% of the total beam length would work best, whereas our results indicate that position is much more influential than size of the actuator for controlling free vibrations.

Finally, it is interesting to note that the apparent filter damping is almost identical to that of the structure even though we specified the true filter damping ratios.

5. APPENDIX

As an aside we note that the limiting factor is using the method described in the **Problem Solution** section is the analytic calculation of the determinant which requires on the order of $n!$ operations for an $n \times n$ matrix. Once we have a polynomial expression in λ , there is no difficulty in determining numerically the roots. We can extend the size of the system for which the above described method is applicable by making use of an identity described by Fortmann [7]. First, we rewrite Γ as:

$$\Gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

When we have the same number of filters as modes, i.e., $N_m = N_f = n$, then A, B, C, D are each $n \times n$ and both A and D are diagonal. The determinant of Γ is then given by [7]:

$$\det \Gamma = |\Gamma| = |A| |D - CA^{-1}B|$$

which is easier to calculate because A is diagonal so both its inversion and determinant calculation are trivial. The number of operations goes from $O((2n)!)$ to $O(n!)$ which may be a great improvement. For example:

$$\begin{aligned} n = 4 & \quad 4! = 24 & \quad 8! = 40320 \\ n = 6 & \quad 6! = 720 & \quad 12! = 479001600 \end{aligned}$$

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NOMENCLATURE

V_s = sensor voltage	V_a = applied voltage	$u(x)$ = unit step function
E = modulus of elasticity	I = moment of inertia	M = applied moment
$\phi_j(x)$ = modeshapes	$z_j(t)$ = generalized coordinates	$\omega(x,t)$ = deflection
δ_{ij} = Kronecker delta	$\mu(x)$ = mass density	C_p = capacitance
d_{31} = piezoceramic charge coefficient	g_{31} = piezoceramic voltage coefficient	Ω_k = diagonal matrix of eigenvalues
L = length	b = width	t = thickness
B = modal participation factor	$n(t)$ = vector of filter coordinates	C = modal participation matrix
G = diagonal matrix of gains	N_m = number of modes	N_f = number of filters