ICSV 17
THE 17th INTERNATIONAL CONGRESS ON SOUND & VIBRATION
CAIRO 18 - 22 JULY, 2010

MULTIBODY DYNAMIC ANALYSIS AND NONLINEAR CONTROL OF FLEXIBLE-MANIPULATORS

Ayman A. Elbadawy, and Mohamed. W. Mehrez

Faculty of Engineering and Material Science, German University in Cairo, Eltagamoaa El Khames, New Cairo, Egypt

e-mail: ayman.elbadawy@guc.edu.eg

In this paper, the analytical formulation of the multibody dynamic (MBD) equations of motion of flexible link manipulators has been developed. The (MBD) equations of motion are in the form of differential algebraic equations (DAEs). These equations are then adapted to suit the application of two control algorithms (feedback linearization and composite control based on singular perturbation theory). The equations structure, to be exploited in the numerical methods, is defined and discussed. The numerical procedure for the solution of the developed DAEs is presented. The DAEs results are compared with the standard Lagrange formulation exploiting ordinary differential equations (ODEs).

1. Introduction

In recent years, great emphasis has been placed on the design of high-speed, lightweight manipulators. Such manipulators can be modelled as a multibody system that consists of deformable bodies. These kinds of multibody systems are called flexible multibody systems. The flexible multibody systems has been the most important research subject in multibody systems, and a great deal of research work has been done on it and many numerical methods have been developed for solving the flexible multibody system dynamic problems, such as the finite element method [1], finite segment method [2], and mode synthesis method [3]. Chen [4] adopted the Lagrange approach to establish equations of motion for multi-link planar flexible manipulators. The rigid-body motion and elastic deformations are decoupled by linearizing the equations of motion around the rigid-body reference trajectory. Leung et al. [5] derived the nonlinear dynamic equations of a multibody system composed of flexible beams by using the Lagrange multiplier method. They used driving constraints to investigate the response of the dynamics model obtained.

In this paper, the flexible multibody system dynamic equations are derived employing Lagrange multiplier method. Feedback linearization control and a composite control based on the singular perturbation approach, which are developed and used solely for ordinary differential equations of motion, are newly developed for the DAE formulation of the equations of motion. The numerical procedure for solving these nonlinear constraint equations are presented based on Newmark direct integration method and Newton-Raphson iterative method. The response of this model is compared with the response obtained by the Lagrange formulation method [6]. These new formulations can be later integrated in commercial Multibody dynamic softwares such as ADAMS® [7].

2. Control of flexible manipulators in MBD form

2.1 Multibody dynamic formulation of the EOM

In order to identify the configuration of deformable manipulators, the system generalized coordinate vector is defined as

$$\mathbf{q} = [\mathbf{q}_r^{1^T} \ \mathbf{q}_r^{2^T} \ldots \mathbf{q}_r^{N^T} \ \mathbf{q}_f^{1^T} \ \mathbf{q}_f^{2^T} \ldots \mathbf{q}_f^{N^T}]^T$$

where N is the total number of bodies in the multibody system. The subscripts r and f refer to rigid and flexible coordinates, respectively. The total kinetic energy and strain energy of the multibody system can be written as

$$T = \sum_{i=1}^{N} T^i = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{M}\dot{\mathbf{q}} \qquad \text{and} \qquad U = \sum_{i=1}^{N} U^i = \frac{1}{2}\mathbf{q}^T \mathbf{K}\,\mathbf{q} \tag{1}$$

Supposing that the system is subjected to a set of holonomic constraints, which can be written in the form

$$\boldsymbol{\varphi}(\mathbf{q}, t) = 0 \tag{2}$$

and utilizing the Lagrange multiplier method for flexible multibody systems [8], flexible manipulator dynamic equations can be written as

$$\mathbf{M}\,\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} + \boldsymbol{\varphi}_{\mathbf{q}}^T \boldsymbol{\lambda} = \mathbf{F} + \mathbf{Q}_v \tag{3}$$

where $\mathbf{M} = \begin{bmatrix} m_{rr} & m_{rf} \\ symmetric & m_{ff} \end{bmatrix}$ is the mass matrix, $\mathbf{K}$ is the stiffness matrix, $\boldsymbol{\varphi}_{\mathbf{q}}$ is the constraint Jacobian matrix, $\boldsymbol{\lambda}$ is the vector of Lagrange multipliers, $\mathbf{F}$ is the generalized external forces vector, and $\mathbf{Q}_v$ is the quadratic velocity vector that contains the gyroscopic and Coriolis forces, which has the form

$$\mathbf{Q}_v = -\dot{\mathbf{M}}\dot{\mathbf{q}} + \frac{1}{2}\frac{\partial}{\partial \mathbf{q}}(\dot{\mathbf{q}}^T \mathbf{M}\dot{\mathbf{q}})^T \tag{4}$$

Eqs. (2 and 3) form the differential algebraic equations that describe the problem.

2.2 MBD feedback linearization based control

In case of rigid multibody systems, the stiffness term in Eq. (3) is cancelled and the dynamics equations of motion can be written in the form

$$\mathbf{M}\ddot{\mathbf{q}} + \boldsymbol{\varphi}_{\mathbf{q}}^{\mathbf{T}}\boldsymbol{\lambda} - \mathbf{Q}_v = \mathbf{F} \tag{5}$$

Here, we seek a nonlinear feedback control law which, when substituted into Eq. (5), results in a linear closed-loop system. If we choose the control vector $\mathbf{F}$ as

$$\mathbf{F} = \mathbf{M}\mathbf{a}_q - \boldsymbol{\varphi}_{\mathbf{q}}^{\mathbf{T}}\boldsymbol{\lambda} + \mathbf{Q}_v \tag{6}$$

Inserting Eq.(6) into Eq.(5) results in

$$\ddot{\mathbf{q}} = \mathbf{a}_q \tag{7}$$

where $\mathbf{a}_q$ can be chosen simply as;

$$\mathbf{a}_q = [\mathbf{0} \quad \mathbf{0} \quad \ddot{\boldsymbol{\theta}}_d + \mathbf{k}_v(\dot{\boldsymbol{\theta}}_d - \dot{\boldsymbol{\theta}}) + \mathbf{k}_p(\boldsymbol{\theta}_d - \boldsymbol{\theta})]^T \tag{8}$$

where $\boldsymbol{\theta}_d, \dot{\boldsymbol{\theta}}_d$, and $\ddot{\boldsymbol{\theta}}_d$ are the desired trajectory position, velocity, and acceleration, respectively. $\mathbf{k}_v$ and $\mathbf{k}_p$ are the velocity and position feedback gain diagonal matrices, respectively.

2.3 MBD composite control based on Singular perturbation approach

In this subsection, a singularly perturbed model for the equations of motion employing Lagrange multipliers is derived. A composite control, based on the derived singularly perturbed model, is then pursued.

The multibody dynamic model for flexible link manipulators Eq. (3) can be rewritten as

$$\mathbf{M}\begin{bmatrix}\ddot{\mathbf{q}}_{r_{n\times1}}\\ \ddot{\mathbf{q}}_{f_{m\times1}}\end{bmatrix}_{(n+m)\times1}+\mathbf{K}\begin{bmatrix}\mathbf{q}_{r_{n\times1}}\\ \mathbf{q}_{f_{m\times1}}\end{bmatrix}_{(n+m)\times1}+\boldsymbol{\varphi}^{\mathbf{T}}_{\mathbf{q}\,(n+m)\times h}\begin{bmatrix}\boldsymbol{\lambda}_{1(h/2)\times1}\\ \boldsymbol{\lambda}_{2(h/2)\times1}\end{bmatrix}_{h\times1}=\begin{bmatrix}\mathbf{F}_{r_{n\times1}}\\ \mathbf{F}_{f_{m\times1}}\end{bmatrix}_{(n+m)\times1}+\begin{bmatrix}\mathbf{Q}_{v\,r_{n\times1}}\\ \mathbf{Q}_{v\,f_{m\times1}}\end{bmatrix}_{(n+m)\times1} \tag{9}$$

where n is the total number of the rigid coordinates, m is the total number of the considered flexible coordinates, and h is the number of the kinematic constraints.

$\mathbf{K}=\begin{bmatrix}\mathbf{0}_{n\times n} & \mathbf{0}\\ \mathbf{0} & \mathbf{K}_{ff\,m\times m}\end{bmatrix}, \mathbf{F}_r=[\mathbf{0}\quad \mathbf{T}^T]^T$ where the vector $\mathbf{T}_{N\times1}$ contains the joint torques, and $\mathbf{F}_f=[\mathbf{0}]$.

Assuming that $\boldsymbol{\varphi}_{\mathbf{q}}$ can be partitioned in the form $\boldsymbol{\varphi}_{\mathbf{q}}=\begin{bmatrix}\boldsymbol{\varphi}_{\mathbf{q}r1(h/2)\times n} & \boldsymbol{\varphi}_{\mathbf{q}f1(h/2)\times m}\\ \boldsymbol{\varphi}_{\mathbf{q}r2(h/2)\times n} & \boldsymbol{\varphi}_{\mathbf{q}f2(h/2)\times m}\end{bmatrix}$, then $\boldsymbol{\varphi}^T_{\mathbf{q}}$ is given by $\boldsymbol{\varphi}^T_{\mathbf{q}}=\begin{bmatrix}\boldsymbol{\varphi}^T_{\mathbf{q}r1} & \boldsymbol{\varphi}^T_{\mathbf{q}r2}\\ \boldsymbol{\varphi}^T_{\mathbf{q}f1} & \boldsymbol{\varphi}^T_{\mathbf{q}f2}\end{bmatrix}$.

Since the mass matrix is invertible, then its inverse is denoted by $\mathbf{P}$ and is given as;

$$\mathbf{M}^{-1}=\mathbf{P}=\begin{bmatrix}\mathbf{P}_{11[n\times n]} & \mathbf{P}_{12[n\times m]}\\ \mathbf{P}_{21[m\times n]} & \mathbf{P}_{22[m\times m]}\end{bmatrix} \tag{10}$$

Now, Eq. (9) can be written in the form

$$\ddot{\mathbf{q}}_r=\mathbf{P}_{11}\mathbf{F}_r+\mathbf{P}_{11}\mathbf{Q}_{vr}+\mathbf{P}_{12}\mathbf{Q}_{vf}-\mathbf{P}_{12}\mathbf{K}_{ff}\mathbf{q}_f-\mathbf{P}_{11}\boldsymbol{\varphi}^T_{\mathbf{q}r1}\boldsymbol{\lambda}_1-\mathbf{P}_{11}\boldsymbol{\varphi}^T_{\mathbf{q}r2}\boldsymbol{\lambda}_2-\mathbf{P}_{12}\boldsymbol{\varphi}^T_{\mathbf{q}f1}\boldsymbol{\lambda}_1-\mathbf{P}_{12}\boldsymbol{\varphi}^T_{\mathbf{q}f2}\boldsymbol{\lambda}_2 \tag{11.a}$$

$$\ddot{\mathbf{q}}_f=\mathbf{P}_{21}\mathbf{F}_r+\mathbf{P}_{21}\mathbf{Q}_{vr}+\mathbf{P}_{22}\mathbf{Q}_{vf}-\mathbf{P}_{22}\mathbf{K}_{ff}\mathbf{q}_f-\mathbf{P}_{21}\boldsymbol{\varphi}^T_{\mathbf{q}r1}\boldsymbol{\lambda}_1-\mathbf{P}_{21}\boldsymbol{\varphi}^T_{\mathbf{q}r2}\boldsymbol{\lambda}_2-\mathbf{P}_{22}\boldsymbol{\varphi}^T_{\mathbf{q}f1}\boldsymbol{\lambda}_1-\mathbf{P}_{22}\boldsymbol{\varphi}^T_{\mathbf{q}f2}\boldsymbol{\lambda}_2 \tag{11.b}$$

To derive the singularly perturbed model of the dynamic system Eq. (9), it is assumed that the orders of magnitude of the $\mathbf{K}_{ff}$ matrix elements ($k_i's$) are comparable. At this step, it is appropriate to extract a common scale factor k (the smallest stiffness term) such that

$$k_i=k\tilde{k}_i,\qquad i=1,\dots,m. \tag{12}$$

The following new variables (elastic forces) can be defined:

$$\boldsymbol{\zeta}=k\,\tilde{\mathbf{k}}_{ff}\boldsymbol{\delta},\qquad \tilde{\mathbf{k}}_{ff}=diag(\tilde{k}_1\cdots\tilde{k}_m) \tag{13}$$

The next step is to define $\mu=1/k$ and obtain

$$\begin{aligned}\ddot{\mathbf{q}}_r=\;&\mathbf{P}_{11}(\mathbf{q}_r,\mu\boldsymbol{\zeta})\mathbf{F}_r+\mathbf{P}_{11}(\mathbf{q}_r,\mu\boldsymbol{\zeta})\mathbf{Q}_{vr}(\mathbf{q}_r,\dot{\mathbf{q}}_r,\mu\boldsymbol{\zeta},\mu\dot{\boldsymbol{\zeta}})\\&+\mathbf{P}_{12}(\mathbf{q}_r,\mu\boldsymbol{\zeta})\mathbf{Q}_{vf}(\mathbf{q}_r,\dot{\mathbf{q}}_r,\mu\boldsymbol{\zeta},\mu\dot{\boldsymbol{\zeta}})-\mathbf{P}_{12}(\mathbf{q}_r,\mu\boldsymbol{\zeta})\mathbf{K}_{ff}\boldsymbol{\zeta}\\&-\mathbf{P}_{11}(\mathbf{q}_r,\mu\boldsymbol{\zeta})\boldsymbol{\varphi}^T_{\mathbf{q}r1}\boldsymbol{\lambda}_1-\mathbf{P}_{11}(\mathbf{q}_r,\mu\boldsymbol{\zeta})\boldsymbol{\varphi}^T_{\mathbf{q}r2}\boldsymbol{\lambda}_2\\&-\mathbf{P}_{12}(\mathbf{q}_r,\mu\boldsymbol{\zeta})\boldsymbol{\varphi}^T_{\mathbf{q}f1}\boldsymbol{\lambda}_1-\mathbf{P}_{12}(\mathbf{q}_r,\mu\boldsymbol{\zeta})\boldsymbol{\varphi}^T_{\mathbf{q}f2}\boldsymbol{\lambda}_2\end{aligned} \tag{14.a}$$

$$\begin{aligned}\mu\ddot{\boldsymbol{\zeta}}=\;&\mathbf{P}_{21}(\mathbf{q}_r,\mu\boldsymbol{\zeta})\mathbf{F}_r+\mathbf{P}_{21}(\mathbf{q}_r,\mu\boldsymbol{\zeta})\mathbf{Q}_{vr}(\mathbf{q}_r,\dot{\mathbf{q}}_r,\mu\boldsymbol{\zeta},\mu\dot{\boldsymbol{\zeta}})\\&+\mathbf{P}_{22}(\mathbf{q}_r,\mu\boldsymbol{\zeta})\mathbf{Q}_{vf}(\mathbf{q}_r,\dot{\mathbf{q}}_r,\mu\boldsymbol{\zeta},\mu\dot{\boldsymbol{\zeta}})-\mathbf{P}_{22}(\mathbf{q}_r,\mu\boldsymbol{\zeta})\mathbf{K}_{ff}\boldsymbol{\zeta}\\&-\mathbf{P}_{21}(\mathbf{q}_r,\mu\boldsymbol{\zeta})\boldsymbol{\varphi}^T_{\mathbf{q}r1}\boldsymbol{\lambda}_1-\mathbf{P}_{21}(\mathbf{q}_r,\mu\boldsymbol{\zeta})\boldsymbol{\varphi}^T_{\mathbf{q}r2}\boldsymbol{\lambda}_2\\&-\mathbf{P}_{22}(\mathbf{q}_r,\mu\boldsymbol{\zeta})\boldsymbol{\varphi}^T_{\mathbf{q}f1}\boldsymbol{\lambda}_1-\mathbf{P}_{22}(\mathbf{q}_r,\mu\boldsymbol{\zeta})\boldsymbol{\varphi}^T_{\mathbf{q}f2}\boldsymbol{\lambda}_2\end{aligned} \tag{14.b}$$

which is a singularly perturbed model of the flexible arm. Notice that all quantities on the right hand side of Eq. (14.b) have been conveniently scaled by $\tilde{\mathbf{k}}_{ff}$. It can be shown that as $\mu\to0$, the model of the rigid manipulator is obtained from Eqs. (14.a and 14.b).

Choosing $\mathbf{w}_1 = \mathbf{q}_r$, $\mathbf{w}_2 = \dot{\mathbf{q}}_r$, and $\mathbf{y}_1 = \boldsymbol{\zeta}$, $\mathbf{y}_2 = \varepsilon\dot{\boldsymbol{\zeta}}$ with $\varepsilon = \sqrt{\mu}$ gives the state-space form of the Eqs. (14.a and 14.b); i.e.

$$
\begin{aligned}
\dot{\mathbf{w}}_1 &= \mathbf{w}_2 \\
\dot{\mathbf{w}}_2 &= \mathbf{P}_{11}(\mathbf{w}_1,\varepsilon^2\mathbf{y}_1)\mathbf{F}_r + \mathbf{P}_{11}(\mathbf{w}_1,\varepsilon^2\mathbf{y}_1)\mathbf{Q}_{vr}(\mathbf{w}_1,\mathbf{w}_2,\varepsilon^2\mathbf{y}_1,\varepsilon\mathbf{y}_2) \\
&\quad + \mathbf{P}_{12}(\mathbf{w}_1,\varepsilon^2\mathbf{y}_1)\mathbf{Q}_{vf}(\mathbf{w}_1,\mathbf{w}_2,\varepsilon^2\mathbf{y}_1,\varepsilon\mathbf{y}_2) - \mathbf{P}_{12}(\mathbf{w}_1,\varepsilon^2\mathbf{y}_1)\mathbf{K}_{ff}\mathbf{y}_1 \\
&\quad - \mathbf{P}_{11}(\mathbf{w}_1,\varepsilon^2\mathbf{y}_1)\boldsymbol{\varphi}_{\mathbf{q}r1}^T\boldsymbol{\lambda}_1 - \mathbf{P}_{11}(\mathbf{w}_1,\varepsilon^2\mathbf{y}_1)\boldsymbol{\varphi}_{\mathbf{q}r2}^T\boldsymbol{\lambda}_2 \\
&\quad - \mathbf{P}_{12}(\mathbf{w}_1,\varepsilon^2\mathbf{y}_1)\boldsymbol{\varphi}_{\mathbf{q}f1}^T\boldsymbol{\lambda}_1 - \mathbf{P}_{12}(\mathbf{w}_1,\varepsilon^2\mathbf{y}_1)\boldsymbol{\varphi}_{\mathbf{q}f2}^T\boldsymbol{\lambda}_2
\end{aligned}
\tag{15.a}
$$

$$
\begin{aligned}
\varepsilon\dot{\mathbf{y}}_1 &= \mathbf{y}_2 \\
\varepsilon\dot{\mathbf{y}}_2 &= \mathbf{P}_{21}(\mathbf{w}_1,\varepsilon^2\mathbf{y}_1)\mathbf{F}_r + \mathbf{P}_{21}(\mathbf{w}_1,\varepsilon^2\mathbf{y}_1)\mathbf{Q}_{vr}(\mathbf{w}_1,\mathbf{w}_2,\varepsilon^2\mathbf{y}_1,\varepsilon\mathbf{y}_2) \\
&\quad + \mathbf{P}_{22}(\mathbf{w}_1,\varepsilon^2\mathbf{y}_1)\mathbf{Q}_{vf}(\mathbf{w}_1,\mathbf{w}_2,\varepsilon^2\mathbf{y}_1,\varepsilon\mathbf{y}_2) - \mathbf{P}_{22}(\mathbf{w}_1,\varepsilon^2\mathbf{y}_1)\mathbf{K}_{ff}\mathbf{y}_1 \\
&\quad - \mathbf{P}_{21}(\mathbf{w}_1,\varepsilon^2\mathbf{y}_1)\boldsymbol{\varphi}_{\mathbf{q}r1}^T\boldsymbol{\lambda}_1 - \mathbf{P}_{21}(\mathbf{w}_1,\varepsilon^2\mathbf{y}_1)\boldsymbol{\varphi}_{\mathbf{q}r2}^T\boldsymbol{\lambda}_2 \\
&\quad - \mathbf{P}_{22}(\mathbf{w}_1,\varepsilon^2\mathbf{y}_1)\boldsymbol{\varphi}_{\mathbf{q}f1}^T\boldsymbol{\lambda}_1 - \mathbf{P}_{22}(\mathbf{w}_1,\varepsilon^2\mathbf{y}_1)\boldsymbol{\varphi}_{\mathbf{q}f2}^T\boldsymbol{\lambda}_2
\end{aligned}
\tag{15.b}
$$

At this point, singular perturbation theory requires that the slow subsystem and the fast subsystem be identified. The slow subsystem is formally obtained by setting $\varepsilon = 0$, i.e., the rigid model of the arm.

$$
\begin{aligned}
\dot{\overline{\mathbf{w}}}_1 &= \overline{\mathbf{w}}_2 \\
\dot{\overline{\mathbf{w}}}_2 &= \mathbf{M}_{rr}^{-1}(\overline{\mathbf{w}}_1)\times[\mathbf{Q}_{vr}(\overline{\mathbf{w}}_1,\overline{\mathbf{w}}_2,\mathbf{0},\mathbf{0}) - \boldsymbol{\varphi}_{\mathbf{q}r1}^T\boldsymbol{\lambda}_1 - \boldsymbol{\varphi}_{\mathbf{q}r2}^T\boldsymbol{\lambda}_2 + \overline{\mathbf{F}}_r]
\end{aligned}
\tag{16}
$$

where $\mathbf{M}_{rr}(\overline{\mathbf{w}}_1)$ is the $n\times n$ positive definite matrix for the rigid-link arm and the over bars are used to indicate that the system with $\varepsilon = 0$ is considered. To derive the fast subsystem, we introduce the fast time scale $\tau = t/\varepsilon$. Then it can be recognized that the system given by Eqs. (15.a and 15.b) in the fast time scale becomes

$$
\begin{aligned}
\frac{d\mathbf{w}_1}{d\tau} &= \varepsilon\mathbf{w}_2 \\
\frac{d\mathbf{w}_2}{d\tau} &= \varepsilon[\mathbf{P}_{11}(\mathbf{w}_1,\varepsilon^2(\boldsymbol{\eta}_1+\overline{\boldsymbol{\zeta}}))\mathbf{F}_r + \mathbf{P}_{11}(\mathbf{w}_1,\varepsilon^2(\boldsymbol{\eta}_1+\overline{\boldsymbol{\zeta}}))\mathbf{Q}_{vr}(\mathbf{w}_1,\mathbf{w}_2,\varepsilon^2(\boldsymbol{\eta}_1+\overline{\boldsymbol{\zeta}}),\varepsilon\boldsymbol{\eta}_2) \\
&\quad + \mathbf{P}_{12}(\mathbf{w}_1,\varepsilon^2(\boldsymbol{\eta}_1+\overline{\boldsymbol{\zeta}}))\mathbf{Q}_{vf}(\mathbf{w}_1,\mathbf{w}_2,\varepsilon^2(\boldsymbol{\eta}_1+\overline{\boldsymbol{\zeta}}),\varepsilon\boldsymbol{\eta}_2) - \mathbf{P}_{12}(\mathbf{w}_1,\varepsilon^2(\boldsymbol{\eta}_1+\overline{\boldsymbol{\zeta}}))\mathbf{K}_{ff}(\boldsymbol{\eta}_1+\overline{\boldsymbol{\zeta}}) \\
&\quad - \mathbf{P}_{11}(\mathbf{w}_1,\varepsilon^2(\boldsymbol{\eta}_1+\overline{\boldsymbol{\zeta}}))\boldsymbol{\varphi}_{\mathbf{q}r1}^T\boldsymbol{\lambda}_1 - \mathbf{P}_{11}(\mathbf{w}_1,\varepsilon^2(\boldsymbol{\eta}_1+\overline{\boldsymbol{\zeta}}))\boldsymbol{\varphi}_{\mathbf{q}r2}^T\boldsymbol{\lambda}_2 \\
&\quad - \mathbf{P}_{12}(\mathbf{w}_1,\varepsilon^2(\boldsymbol{\eta}_1+\overline{\boldsymbol{\zeta}}))\boldsymbol{\varphi}_{\mathbf{q}f1}^T\boldsymbol{\lambda}_1 - \mathbf{P}_{12}(\mathbf{w}_1,\varepsilon^2(\boldsymbol{\eta}_1+\overline{\boldsymbol{\zeta}}))\boldsymbol{\varphi}_{\mathbf{q}f2}^T\boldsymbol{\lambda}_2]
\end{aligned}
\tag{17.a}
$$

$$
\begin{aligned}
\frac{d\boldsymbol{\eta}_1}{d\tau} &= \boldsymbol{\eta}_1 \\
\frac{d\boldsymbol{\eta}_2}{d\tau} &= \mathbf{P}_{21}(\mathbf{w}_1,\varepsilon^2(\boldsymbol{\eta}_1+\overline{\boldsymbol{\zeta}}))\mathbf{F}_r + \mathbf{P}_{21}(\mathbf{w}_1,\varepsilon^2(\boldsymbol{\eta}_1+\overline{\boldsymbol{\zeta}}))\mathbf{Q}_{vr}(\mathbf{w}_1,\mathbf{w}_2,\varepsilon^2(\boldsymbol{\eta}_1+\overline{\boldsymbol{\zeta}}),\varepsilon\boldsymbol{\eta}_2) \\
&\quad + \mathbf{P}_{22}(\mathbf{w}_1,\varepsilon^2(\boldsymbol{\eta}_1+\overline{\boldsymbol{\zeta}}))\mathbf{Q}_{vf}(\mathbf{w}_1,\mathbf{w}_2,\varepsilon^2(\boldsymbol{\eta}_1+\overline{\boldsymbol{\zeta}}),\varepsilon\boldsymbol{\eta}_2) - \mathbf{P}_{22}(\mathbf{w}_1,\varepsilon^2(\boldsymbol{\eta}_1+\overline{\boldsymbol{\zeta}}))\mathbf{K}_{ff}(\boldsymbol{\eta}_1+\overline{\boldsymbol{\zeta}}) \\
&\quad - \mathbf{P}_{21}(\mathbf{w}_1,\varepsilon^2(\boldsymbol{\eta}_1+\overline{\boldsymbol{\zeta}}))\boldsymbol{\varphi}_{\mathbf{q}r1}^T\boldsymbol{\lambda}_1 - \mathbf{P}_{21}(\mathbf{w}_1,\varepsilon^2(\boldsymbol{\eta}_1+\overline{\boldsymbol{\zeta}}))\boldsymbol{\varphi}_{\mathbf{q}r2}^T\boldsymbol{\lambda}_2 \\
&\quad - \mathbf{P}_{22}(\mathbf{w}_1,\varepsilon^2(\boldsymbol{\eta}_1+\overline{\boldsymbol{\zeta}}))\boldsymbol{\varphi}_{\mathbf{q}f1}^T\boldsymbol{\lambda}_1 - \mathbf{P}_{22}(\mathbf{w}_1,\varepsilon^2(\boldsymbol{\eta}_1+\overline{\boldsymbol{\zeta}}))\boldsymbol{\varphi}_{\mathbf{q}f2}^T\boldsymbol{\lambda}_2
\end{aligned}
\tag{17.b}
$$

where the new fast variables $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ are defined as

$$
\boldsymbol{\eta}_1 = \mathbf{y}_1 - \overline{\boldsymbol{\zeta}}, \qquad \boldsymbol{\eta}_2 = \mathbf{y}_2 \tag{18}
$$

$\overline{\boldsymbol{\zeta}}$ is obtained from Eq.(14.b) after setting $\mu = 0$

Now setting $\varepsilon = 0$ in Eqs. (17.a and 17.b) gives $\frac{d\mathbf{w}_1}{d\tau} = \frac{d\mathbf{w}_2}{d\tau} = 0$. And the fast subsystem can be found to be

$$\frac{d\boldsymbol{\eta}_1}{d\tau} = \boldsymbol{\eta}_2 ,$$
$$\frac{d\boldsymbol{\eta}_2}{d\tau} = -\mathbf{P}_{22}(\overline{\mathbf{w}}_1,0)\boldsymbol{\eta}_1 + \mathbf{P}_{21}(\overline{\mathbf{x}}_1,0)(\mathbf{F}_r - \overline{\mathbf{F}}_r) \tag{19}$$

where $(\mathbf{F}_r - \overline{\mathbf{F}}_r) = \begin{bmatrix} \mathbf{0} \\ \mathbf{T} \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ \overline{\mathbf{T}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{T}_f \end{bmatrix}$. It is clear that the system (19) is a linear system parameterized in the slow variables $\overline{\mathbf{w}}_1$.

As evidenced by the two subsystems (16) and (19), a composite control strategy [9] can be pursued. The design of a feedback control for the full system can be split into two separated designs of feedback controls $\overline{\mathbf{T}}$ and $\mathbf{T}_f$ for the two reduced-order systems; formally

$$\mathbf{T} = \overline{\mathbf{T}}(\overline{\mathbf{w}}_1, \overline{\mathbf{w}}_2) + \mathbf{T}_f(\overline{\mathbf{w}}_1, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) \tag{20}$$

where $\overline{\mathbf{T}}(\overline{\mathbf{w}}_1, \overline{\mathbf{w}}_2)$ is the slow control and $\mathbf{T}_f(\overline{\mathbf{w}}_1, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ is the fast control.

As far as the slow control is concerned, the feedback linearization based control, developed for rigid manipulators, can be utilized. Then, the fast control can be designed as an optimal control.

3. Numerical Procedure

The response of the Multibody dynamics model can be solved by the Newmark direct integration algorithm [10] and Newton-Raphson iterative method [11] presented here. The numerical procedures are as follows:

(1) Convert the differential algebraic equations into algebraic nonlinear equations using Newmark's formulas Eq. (21) that express the generalized position and velocity as functions of generalized accelerations at t_{n+1}.

$$\mathbf{q}_{n+1} = \mathbf{q}_n + h\dot{\mathbf{q}}_n + \frac{h^2}{2}[(1-2\beta)\ddot{\mathbf{q}}_n + 2\beta\ddot{\mathbf{q}}_{n+1}]$$
$$\dot{\mathbf{q}}_{n+1} = \dot{\mathbf{q}}_n + h[(1-\gamma)\ddot{\mathbf{q}}_n + \gamma\ddot{\mathbf{q}}_{n+1}] \tag{21}$$

(2) Find acceleration and Lagrange multipliers, using Newton-Raphson iterative method, that satisfy

$$\boldsymbol{\Psi}(\ddot{\mathbf{q}}_{n+1}, \boldsymbol{\lambda}_{n+1}) = \begin{bmatrix} \mathbf{M}(\mathbf{q}_{n+1})\ddot{\mathbf{q}}_{n+1} + \boldsymbol{\varphi}_{\mathbf{q}}^T(\mathbf{q}_{n+1})\boldsymbol{\lambda}_{n+1} + \mathbf{K}\mathbf{q}_{n+1} - \mathbf{Q}_v(\mathbf{q}_{n+1}, \dot{\mathbf{q}}_{n+1}) - \mathbf{F}(t_{n+1}, \mathbf{q}_{n+1}, \dot{\mathbf{q}}_{n+1}) \\ \boldsymbol{\varphi}(\mathbf{q}_{n+1}, t_{n+1}) \end{bmatrix} = 0 \tag{22}$$

This calls for the Jacobian of the nonlinear system of equations (chain rule will be used to simplify calculations). Based on Newmark integrations formulas, the Jacobian is calculated as:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial \boldsymbol{\Psi}}{\partial \ddot{\mathbf{q}}} & \frac{\partial \boldsymbol{\Psi}}{\partial \boldsymbol{\lambda}} \end{bmatrix} = \left[\begin{array}{c|c} \mathrm{T}_1 + \mathrm{T}_2 + \mathrm{T}_3 - \mathrm{T}_4 - \mathrm{T}_5 & \boldsymbol{\varphi}_{\mathbf{q}}^T \\ \hline \beta h^2 \boldsymbol{\varphi}_{\mathbf{q}} & \mathbf{0} \end{array}\right] \tag{23}$$

where $\mathrm{T}_1 = \beta h^2 \frac{\partial}{\partial \mathbf{q}}[\mathbf{M}\ddot{\mathbf{q}}_{n+1}] + \mathbf{M}$, $\mathrm{T}_2 = \beta h^2 \frac{\partial}{\partial \mathbf{q}}[\boldsymbol{\varphi}_{\mathbf{q}}^T \boldsymbol{\lambda}]$, $\mathrm{T}_3 = \beta h^2 \mathbf{K}$

$$\mathrm{T}_4 = \beta h^2 \frac{\partial}{\partial \mathbf{q}}[\mathbf{Q}_v] + \gamma h \frac{\partial}{\partial \dot{\mathbf{q}}}[\mathbf{Q}_v], \ \mathrm{T}_5 = \beta h^2 \frac{\partial}{\partial \mathbf{q}}[\mathbf{F}] + \gamma h \frac{\partial}{\partial \dot{\mathbf{q}}}[\mathbf{F}] \tag{24}$$

And the iteration equation will be

$$\begin{bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{bmatrix}^{(new)} = \begin{bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{bmatrix}^{(old)} - \begin{bmatrix} \delta\ddot{\mathbf{q}} \\ \delta\boldsymbol{\lambda} \end{bmatrix} \tag{25}$$

where the correction term is computed through.

$$\begin{bmatrix} \delta \ddot{\mathbf{q}} \\ \delta \boldsymbol{\lambda} \end{bmatrix} = \mathbf{J}^{-1} \, \boldsymbol{\Psi}(\ddot{\mathbf{q}}^{(old)}, \boldsymbol{\lambda}^{(old)}) \tag{26}$$

Note: subscripts "$n+1$" are dropped to keep the presentation simple. The iteration procedure will continue until the following convergence condition is satisfied

$$\left\| \begin{bmatrix} \delta \ddot{\mathbf{q}} \\ \delta \boldsymbol{\lambda} \end{bmatrix} \right\| < e \text{ where } e \text{ is a predefined tolerance.}$$

(3) Compute the position and velocity at time step t_{n+1} using Eq. (21)

4. Numerical Example

To investigate the performance of the developed control methods, a study on a one-link flexible arm shown in Fig.1 that is moving in the horizontal plane is performed. The response of the multibody dynamics model of the arm, under the two control techniques, is compared with the response of the Lagrange dynamics model of the arm. The first two deformation modes of the arm are considered (m = 2) since the higher modes are observed to be negligible. The arm is commanded with a step input change from θ = 0° to θ = 90°. The simulation results are presented in two sets. The first set shows the response of the arm under the feedback linearization based control technique, while the second set shows the arm response under the composite control. The gains of the feedback linearization based control are obtained based on the smallest natural frequency of the flexible arm [12]. The fast control is chosen according to the Linear Quadratic Regulator **LQR** technique [13], where the required performance measures are achieved by choosing the weighing matrices **Q** and **R** as **Q** = 35**I** and **R** = 8**I**.

The data of the beam used are beam mass M = 0.2 Kg, payload mass M_p = 0.1 Kg, payload inertia J_p = 0.001 Kg. m^2, joint inertia J_h = 0.5 Kg. m^2, beam length L=1.2 m, and flexural rigidity EI = 60 N.m^2.

Under the feedback linearization based control, Figs. (2 and 3) show the first and second modal variables response, respectively. The first modal variable response, based on Lagrange formulation, is realized to be very close to that based on the multibody dynamics formulation, while a discrepancy in response for the second modal variable is noticed. However, the effect of the second mode can be neglected since the order of its magnitude is in the range of less than tenth of a mm. Figs. (4, and 5) show the control torque used, and the joint variable response, respectively, where the simulation results under both simulations presented are nearly the same.

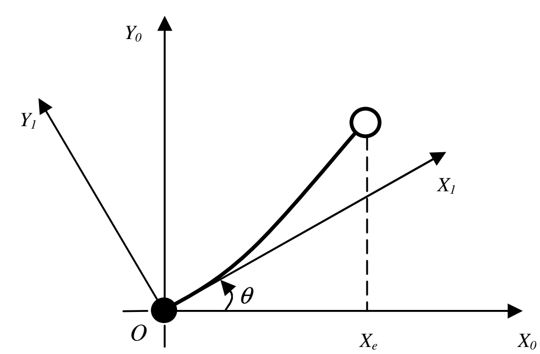

Figure 1. One-link flexible arm

Under the singular perturbation based control, Figs. (6 and 7) show the first and second modal variables response, respectively, where the response of the two variables is improved compared with their response under the feedback linearization based control. Also, a small difference between the MBD formulation based response and the Lagrange formulation based response is realized. Figs. (8, and 9) show the control torque used, and the joint variable response, respectively.

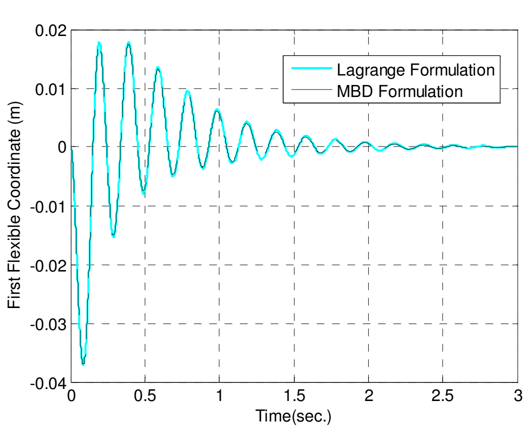

Figure 2. First modal variable (feedback linearization based control)

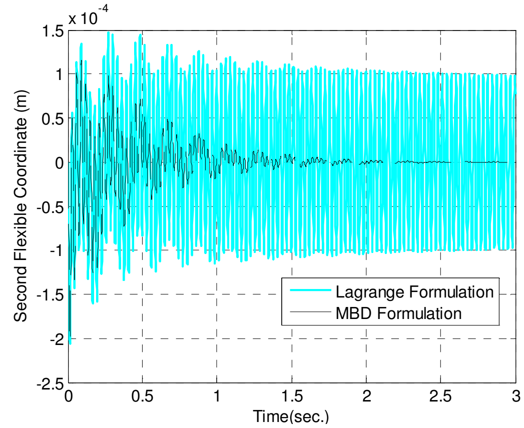

Figure 3. second modal variable (feedback linearization based control)

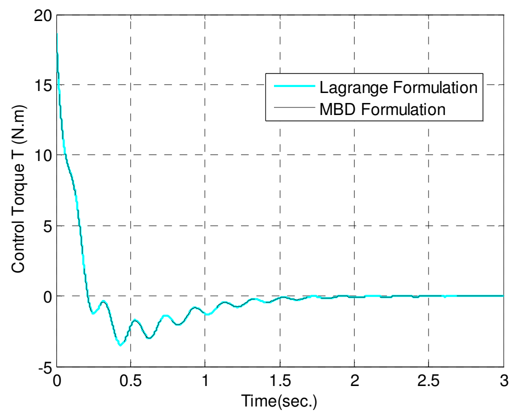

Figure 4. Control Torque (feedback linearization based control)

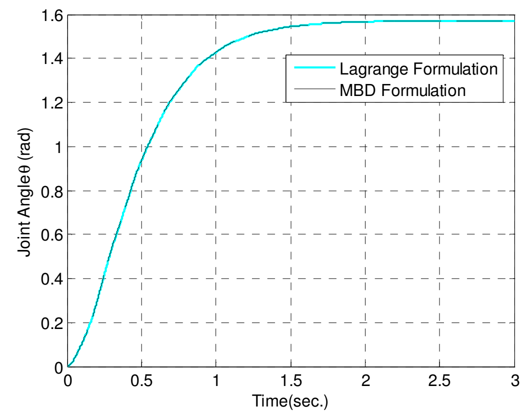

Figure 5. Joint angle (feedback linearization based control)

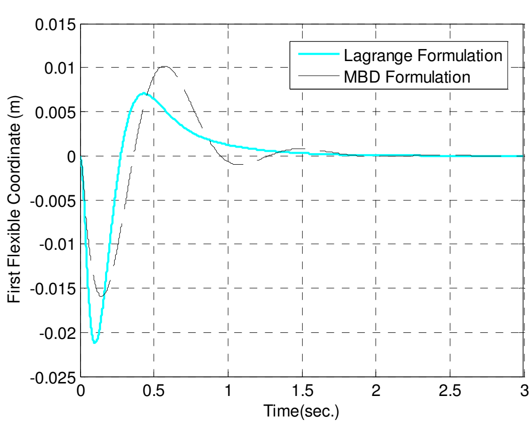

Figure 6. First modal variable (Singular perturbation based control)

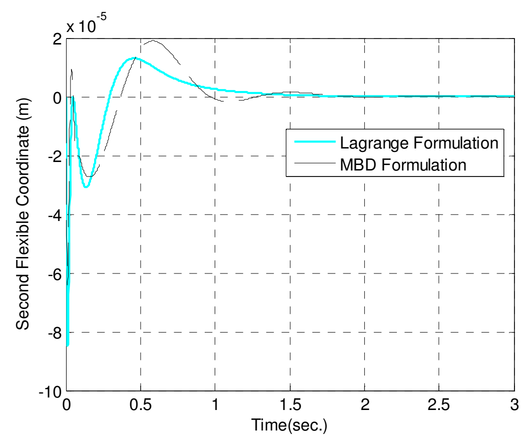

Figure 7. second modal variable (Singular perturbation based control)

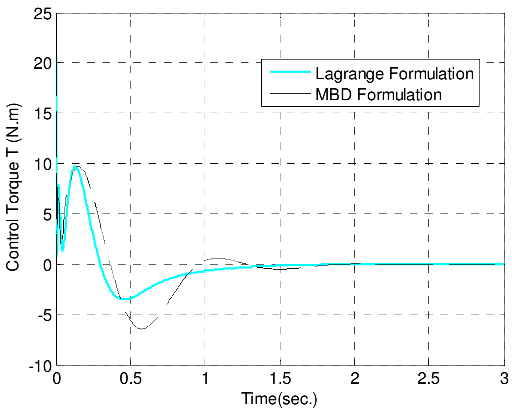

Figure 8. Control Torque (Singular perturbation based control)

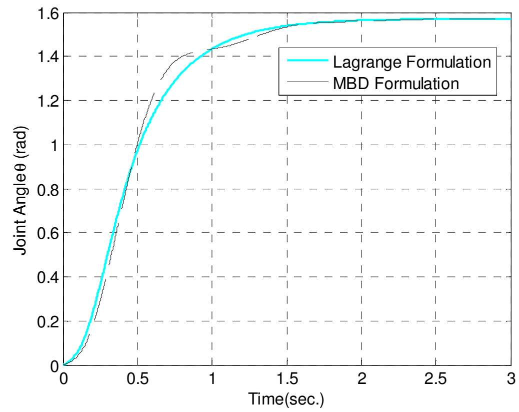

Figure 9. Joint angle (Singular perturbation based control)

The similarity of the results between the two techniques of integration (Runge-Kutta for the case of Lagrange formulation and Newmark-Newton Raphson for MBD) can be improved if different integration technique such as Hilber-Hughes-Taylor [14] is used.

5. Conclusion

In this work, feedback linearization and singular perturbation based control techniques are newly applied on a developed equations of motion utilizing the multibody dynamic formulation. A numerical example was carried on a single link flexible arm. The response of the MBD model under the developed control algorithms are compared with that developed using Lagrange formulation. The results of both simulations show that they are almost identical. The benefit of the new formulation would be on its ability to be integrated with commercial software to perform control on a broad range of flexible multibody systems.

References

[1] H. D., M. K.L., K. M.L., A nonlinear finite element model for dynamics of flexible manipulators, *Mechanism and Machine Theory*, Vol. 31, 1996, pp. 1109-1119.
[2] J. D. Connelly and R. L. Huston, The dynamics of flexible multibody systems: A finite segment approach—I. Theoretical aspects, *Computers and Structures*, Vol. 50, 1994, pp. 255-258.
[3] J. Ryu, H. S. Kimag and S. Wangag , A method for improving dynamic solutions in flexible multibody dynamics, *Computers and Structures*, Vol. 79, 2001, pp.183-195.
[4] Wen Chen, Dynamic modeling of multi-link flexible robotic manipulators, *Computers and Structures*, Vol. 66, 1998, pp. 765-776.
[5] A. Y. T. Leung, Wu Guorong, Zhong Weifang, Nonlinear Dynamic Analysis of Flexible Multibody System, *Acta Mechanica Solida Sinica*, Vol. 17, 2004, pp. 330-336.
[6] M.W.Mehrez, A.A. El-Badawy, Effect of the joint inertia on selection of under-actuated control algorithm for flexible-link manipulators, *Mechanism and Machine Theory (2010)*, doi:10.1016/j.mechmachtheory.2010.03.003
[7] http://www.mscsoftware.com/Contents/Products/CAE-Tools/Adams.aspx
[8] A. A. Shabana, *Dynamics of Multibody Systems*, Cambridge University Press, 2005.
[9] P. Kokotovic, H. K. Khalil, J. O'Reilly, *Singular Perturbation in Control*, SIAM, 1999
[10] Newmark, N. M., A Method of Computation for Structural Dynamics, ASCE *Journal of the Engineering Mechanics Division*, Vol. 85, 1959, pp.67-94.
[11] S. C. Chapra, R. P. Canale, *Numerical Methods for Engineers*, Fifth Edition, MaGraw-Hill, 2006.
[12] L. Meirovitch, *Fundamentals of Vibrations*, McGraw Hill, 2001.
[13] K. Ogata, *Modern Control Engineering*, Prentice-Hall, 2002.
[14] H.M. Hilber, T.J.R. Hughes, and R. L. Taylor, Improved numerical dissipation for time integration algorithms in structural dynamics, *Earthquake Eng. And Struct. Dynamics*, Vol. 5, 1977, pp. 283-292