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THE USE OF RATIONAL POLYNOMIALS AND ORTHOGONAL POLYNOMIALS FOR FREQUENCY-RESPONSE CURVE FITTING

A. A. El-Badawy

Department of Mechanical Engineering, Al-Azhar University

ABSTRACT

Two techniques that estimate frequency-response parameters are compared and contrasted. Both techniques produce the same result, thus are redundant. The first method assumes a rational polynomial form of solution. The second method uses series of orthogonal polynomials. The denominator allows *n* degrees of freedom and the numerator allows *m* degrees of freedom. In a least-squared error sense, the rational polynomial method requires the inversion of a $n+m+1 \times n+m+1$ matrix. This inversion is a computationally complex operation for significant *n* and *m*. However, the orthogonality of the second method's polynomial reduces the computational complexity since it exhibit symmetries that break the matrix inversion into two smaller matrix inversions thus reducing the computational complexity

KEYWORDS: Rational Polynomials, Orthogonal Polynomials, Parameter Estimation, Frequency Response.

1. INTRODUCTION

This paper presents two techniques that estimate frequency response parameters. The first method assumes a rational polynomial form of solution. The denominator allows *n* degrees of freedom and the numerator allows *m* degrees of freedom. The second method uses series of orthogonal polynomials. The highest order polynomial in the denominator is *n* and the highest order polynomial in the denominator is *m*. Both techniques produce the same result, thus are redundant. However, the orthogonality of the second's polynomials reduces the computational complexity. From the resulting polynomials, the damping, frequency, amplitude and phase information can be found. The roots of the denominator estimate the damping and frequency. The corresponding roots of the numerator estimate amplitude and phase.

Both techniques identify zeros and poles. As the numerator approaches zero, a corresponding zero is approached and as the denominator approaches a zero, a corresponding pole is approached. An error function, $\{e\}$ will be established that will explain how the data is being fit. The error function is formulated to permit computational efficiency. Once the error function has been chosen, there are several criteria possible for minimizing that error. The two most common methods are: (*i*) minimize the maximum error, (*ii*) minimize the sum of squared errors. Mathematical expedience favors criteria (*ii*). Since the error function is a column, an objective function, $O=\left\{e^T\right\}\left\{e\right\}$ will represent error as a real positive number (the superscript T denotes the conjugate transpose). This objective also gives weight to all of the data, not just a single extreme data point. In a least squares sense, the best estimate is determined by the following procedure. The rational polynomial method requires the inversion of a $n+m+1\times n+m+1$ matrix. This inversion is a computationally complex operation, for significant n and m. The use of an orthogonal polynomial set requires special coding considerations. The orthogonal polynomial must be developed and decomposed. However, the orthogonal polynomial method is more computationally efficient, for large m and n. Orthogonal polynomials exhibit symmetries that break the matrix inversion into two smaller matrix inversions thus reducing the computational complexity.

The following sections are descriptions and derivations of the two curve fitting techniques. These techniques assume linear and independent resonances. Small response values, where the signal to noise ratio is high, should be avoided, since these schemes fit all data with equal weight. The error formulation may tend to exaggerate small values from what would be expected by a visual inspection. Parameter estimation should be calculated from data near resonance with as much resolution as possible.

2. RATIONAL FRACTION POLYNOMIAL FORMULATION [1]:

The form of the transfer function is taken as:

$$H(\omega)=\sum_{\frac{n}{2}}\frac{r_k}{S-P_k}+\frac{r_k^*}{S-P_k}$$

Such that,

$$H\left(\omega\right)=\frac{\sum_{m} a_k S^k}{\sum_{n} b_k S^k}$$

$$P_k=\sigma_k+i\omega_k \qquad \text{i.e. the k}^{\text{th}}\text{pole}$$

$$P_k^*=\sigma_k-i\omega_k \qquad \text{i.e. the complex conjugate of the k}^{\text{th}}\text{pole}$$

σ_k is damping and ω_k is the resonant frequency. r_k and r_k^* are the residues for the kth poles. The value of m can be increased to facilitate a better fit which will result in a remainder being generated. In this case, the remainder is a polynomial of order *m-n*.

The experimentally obtained transfer function is represented as h_j. Take,

$$h_j \cong H(\omega)$$

And substitute,

$$h_j \cong \frac{\sum_m a_k S^k}{\sum_n b_k S^k}$$

The error function could be chosen as the difference between the estimated and experimental transfer functions:

$$h_j \cdot \sum_n b_k S^k \cong \sum_m a_k S^k$$

Taking the error function as,

$$\{e\} \equiv \sum_m a_k S^k - h_j \cdot \sum_n b_k S^k$$

is more computationally efficient. This definition quickly tends to a solution of this problem. There are difficulties that are introduced with this definition of error, namely for small response value where signal-to-noise is high. By setting b_n equal to one, the error function becomes,

$$\{e\} = \sum_m a_k S^k - h_j \cdot \left[\sum_{n-1} b_k S^k + (i\omega)^n \right]$$

Setting b_n equal to one places the determination of amplitude in the numerator. This reduces the number of equations from $n+m+2$ to $n+m+1$. Upon close examination, this approach is consistent with the observation that there exists at most $n+m+1$ independent relations.

Define,

$$[P] \equiv \begin{bmatrix} 1 & j\omega_1 & (j\omega_1)^2 & \cdots & (j\omega_1)^m \\ 1 & j\omega_2 & (j\omega_2)^2 & \cdots & (j\omega_2)^m \\ 1 & j\omega_3 & (j\omega_3)^2 & \cdots & (j\omega_3)^m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & j\omega_L & (j\omega_L)^2 & \cdots & (j\omega_L)^m \end{bmatrix}; T \equiv \begin{bmatrix} h_1 & h_1 j\omega_1 & h_1(j\omega_1)^2 & \cdots & h_1(j\omega_1)^{n-1} \\ h_2 & h_2 j\omega_2 & h_2(j\omega_2)^2 & \cdots & h_2(j\omega_2)^{n-1} \\ h_3 & h_3 j\omega_3 & h_3(j\omega_3)^2 & \cdots & h_3(j\omega_3)^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_L & h_L j\omega_L & h_L(j\omega_L)^2 & \cdots & h_L(j\omega_L)^{n-1} \end{bmatrix}$$

$$\{A\} = \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_m \end{Bmatrix}; \{B\} = \begin{Bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{Bmatrix}; \{W\} \equiv \begin{Bmatrix} h_1(j\omega_1)^n \\ h_2(j\omega_1)^n \\ h_3(j\omega_1)^n \\ \vdots \\ h_L(j\omega_1)^n \end{Bmatrix}$$

Observe that the error function can now be expressed in matrix form,

$$\{e\} = [P]\{A\} - [T]\{B\} - \{W\}$$
$$\{e^T\} = \{A^T\}\left[P^T\right] - \{B^T\}\left[T^T\right] - \{W^T\}$$

The sum of squared error objective function is defined as,

$$O \equiv \{e^T\}\{e\}$$

Substituting

$$O = \{A^T\}\left[P^T\right][P]\{A\} - 2\cdot\{A^T\}\left[P^T\right][T]\{B\} - 2\cdot\{A^T\}\left[P^T\right]\{W\} +$$
$$\{B^T\}\left[T^T\right][T]\{B\} - 2\cdot\{B^T\}\left[T^T\right]\{W\} + \{W^T\}\{W\}$$

This function is a minimum when $\partial O/\partial A$ and $\partial O/\partial B$ are zero. The solution of the problem rest on resolving the following relations; by determining appropriate $\{A\}$ and $\{B\}$. This can be accomplished when $[P], [T]$ and $\{W\}$ are known.

$$2\cdot\left[P^T\right][P]\{A\} - 2\cdot\left[P^T\right][T]\{B\} - 2\cdot\left[P^T\right]\{W\} = 0$$
$$-2\cdot\left[T^T\right][P]\{A\} + 2\cdot\left[T^T\right][T]\{B\} - 2\cdot\left[T^T\right]\{W\} = 0$$

Note that this can be reassembled in the form,

$$\left[\begin{array}{cc}\left[P^T\right][P] & -\left[P^T\right][T] \\ \hline -\left[T^T\right][P] & \left[T^T\right][T]\end{array}\right]\left\{\begin{array}{c}A \\ \hline B\end{array}\right\} = \left\{\begin{array}{c}\left[P^T\right]\{W\} \\ \hline \left[T^T\right]\{W\}\end{array}\right\}$$

The values of $\{A\}$ and $\{B\}$ can be determined by a matrix inversion.

3. REFORMULATION USING FORSYTHE POLYNOMIALS [2]:

The advantage of using an orthogonal set of polynomials is that the matrices $\left[P^T\right][P]$ and $\left[T^T\right][T]$ can be diagonalized with zero entries on the off diagonals. Therefore their matrix inversion is merely an algebraic manipulation of the main diagonal. It is possible to define Forsythe polynomials such that $\left[P^T\right][P]$ and $\left[T^T\right][T]$ are identity matrices. This procedure was not executed in the work presented here.

3.1 Generation of Forsythe Orthogonal Polynomial:

Set the polynomial, $P_0(x_\mu)$ equals to one. By definition $P_1(x_\mu)$ is orthogonal to $P_0(x_\mu)$ if, and only if,

$$\sum_\mu P_1(x_\mu)P_0(x_\mu) \equiv 0$$

The polynomial $P_1(x_\mu)$ is taken to have the form

$$xP_0(x_\mu) - \alpha_1 P_0(x_\mu)$$

Where x_μ is a set of control parameters. In frequency analysis x_μ is a collection of frequencies. The x_μ's need not be evenly spaced. Therefore,

$$\sum_\mu P_1(x_\mu) P_0(x_\mu) = \sum_\mu x_\mu P_0(x_\mu) P_0(x_\mu) - \alpha_1 \sum_\mu P_0(x_\mu) = 0$$

Demanding,

$$\alpha_1 = \frac{\sum_\mu x_\mu P_0(x_\mu) P_0(x_\mu)}{\sum_\mu P_0(x_\mu) \sum_\mu P_0(x_\mu)}$$

Similarly, the polynomial $P_2(x_\mu)$ is taken to have the form

$$xP_1(x_\mu) - \alpha_2 P_1(x_\mu) - \beta_1 P_0(x_\mu)$$

Therefore,

$$\sum_\mu P_2(x_\mu) P_1(x_\mu) = \sum_\mu x_\mu P_1(x_\mu) P_1(x_\mu) - \alpha_2 \sum_\mu P_1(x_\mu) \sum_\mu P_1(x_\mu) - \beta_1 \sum_\mu P_0(x_\mu) \sum_\mu P_1(x_\mu) = 0$$

$$\sum_\mu P_2(x_\mu) P_0(x_\mu) = \sum_\mu x_\mu P_1(x_\mu) P_0(x_\mu) - \alpha_2 \sum_\mu P_1(x_\mu) \sum_\mu P_0(x_\mu) - \beta_1 \sum_\mu P_0(x_\mu) \sum_\mu P_0(x_\mu) = 0$$

Demanding,

$$\alpha_2 = \frac{\sum_\mu x_\mu P_1(x_\mu) P_1(x_\mu)}{\sum_\mu P_1(x_\mu) P_1(x_\mu)}$$

$$\beta_1 = \frac{\sum_\mu x_\mu P_1(x_\mu) P_0(x_\mu)}{\sum_\mu P_0(x_\mu) P_0(x_\mu)}$$

In general form,

$$P_{i+1}(x_\mu) \equiv xP_i(x_\mu) - \alpha_{i+1} P_i(x_\mu) - \beta_i P_{i-1}(x_\mu)$$

Where,

$$\sum_\mu P_{i+1}(x_\mu) P_{i-1}(x_\mu) = \sum_\mu x_\mu P_i(x_\mu) P_{i-1}(x_\mu) - \alpha_{i+1} \sum_\mu P_i(x_\mu) \sum_\mu P_{i-1}(x_\mu) - \beta_i \sum_\mu P_{i-1}(x_\mu) \sum_\mu P_{i-1}(x_\mu) = 0$$

$$\sum_\mu P_{i+1}(x_\mu) P_i(x_\mu) = \sum_\mu x_\mu P_i(x_\mu) P_i(x_\mu) - \alpha_{i+1} \sum_\mu P_i(x_\mu) \sum_\mu P_i(x_\mu) - \beta_i \sum_\mu P_{i-1}(x_\mu) \sum_\mu P_i(x_\mu) = 0$$

Giving the recursive relationship for α_{i+1} and β_i,

$$\alpha_{i+1} = \frac{\sum_{\mu} x_\mu P_i\left(x_\mu\right) P_i\left(x_\mu\right)}{\sum_{\mu} P_i\left(x_\mu\right) P_i\left(x_\mu\right)}$$

$$\beta_i = \frac{\sum_{\mu} x_\mu P_i\left(x_\mu\right) P_{i-1}\left(x_\mu\right)}{\sum_{\mu} P_{i-1}\left(x_\mu\right) P_{i-1}\left(x_\mu\right)}$$

This last series of relationships generate and decompose the Forsythe polynomials.

3.2 Forsythe Orthogonal Polynomial Formulation:
Recall that,

$$2\cdot\left[P^T\right][P]\{A\}-2\cdot\left[P^T\right][T]\{B\}-2\cdot\left[P^T\right]\{W\}=0$$
$$-2\cdot\left[T^T\right][P]\{A\}+2\cdot\left[T^T\right][T]\{B\}-2\cdot\left[T^T\right]\{W\}=0$$

The *P*'s represent the Forsythe Polynomials, rather than ordinary polynomial terms as before. $[T]$ and $\{W\}$ require appropriate iteration. The developments of the previous equations are the same for the rational fraction polynomial approach, as is for the orthogonal polynomial approach. However, notice that

$$\left(\left[P^T\right][P]\right)^{-1}\left[P^T\right][P]\{A\}=\left(\left[P^T\right][P]\right)^{-1}\left[P^T\right][T]\{B\}+\left(\left[P^T\right][P]\right)^{-1}\left[P^T\right]\{W\}$$
$$\left(\left[T^T\right][T]\right)^{-1}\left[T^T\right][T]\{B\}=-\left(\left[T^T\right][T]\right)^{-1}\left[P^T\right][T]\{A\}+\left(\left[T^T\right][T]\right)^{-1}\left[T^T\right]\{W\}$$

Where $\left(\left[T^T\right][T]\right)^{-1}$ and $\left(\left[P^T\right][P]\right)^{-1}$ are inexpensive calculations because off-diagonal entries are zero by definition of orthogonality,

$$\sum_{\mu} P_i\left(x_\mu\right) P_j\left(x_\mu\right) \equiv 0, i \neq j$$

Reducing to,

$$\{A\}=\left(\left[P^T\right][P]\right)^{-1}\left[P^T\right][T]\{B\}+\left(\left[P^T\right][P]\right)^{-1}\left[P^T\right]\{W\}$$
$$\{B\}=\left(\left[T^T\right][T]\right)^{-1}\left[T^T\right][P]\{A\}+\left(\left[T^T\right][T]\right)^{-1}\left[T^T\right]\{W\}$$

These relations for determining $\{A\}$ and $\{B\}$ can be expressed independently, hence breaking the matrix inversion into two smaller matrix inversions and thus reducing the computational complexity

RESULTS
Following are some results from the use of the two techniques for fitting a frequency response function (FRF) using a polynomial fit. In all figures, the circles represent the samples of the original FRF and the solid line the formed fit.
Figure 1 uses the rational fraction polynomial (RFP) technique which does not make use of orthogonal functions to calculate the FRF fit. This results in more lengthy calculations in performing the fit. The

final result is acceptable, with the peaks in the fit corresponding to the peaks in the original FRF although the zeros are not quiet found correctly. Figure 2 uses the rational fraction orthogonal polynomial technique (RFOP) which does make use of orthogonal functions to calculate the FRF fit. This results in reduced number of calculations due to the characteristic of the final mathematical operation of inverting the matrix to determine the FRF fit. This last matrix can be inverted in two steps, each a simpler inversion instead of one large matrix inversion, which is computationally intensive.

If the orders of the numerator and denominator polynomials are reduced, then the fit becomes fairly poor, because the order of the fit is now less than the order of the FRF trying to be fit and the polynomial fit cannot accurately reach all of the poles and zeros. Figures 3 and 4 show that the only pole and zero, (for both techniques) that is correctly located, are the ones corresponding to the lowest frequencies, but the magnitudes are still not quite correctly fit.

The correct order for the numerator and denominator will need to be found experimentally, to the point that the fit is qualitatively correct. If the order is overestimated, then the FRF is still fit well, but with some extra poles and zeros brought into the system that may or may not be accurately present. Figure 5 shows there are several poles and zeros that don't actually exist, but that the fit shows exist. Rational Fraction Orthogonal Polynomial technique produces similar results for the same case.

The overestimation of the order results in the fit over stating discrepancies in the data. These slight discrepancies could be noise, or actual small poles and zeros in the system. As stated earlier, all data is treated equally, with no attention paid for whether the data is noisy or not. The increased order makes the model fit more sensitive to small changes in the original data.

CONCLUSIONS

The use of orthogonal polynomials to fit the FRF does provide a reduction in the calculation complexity needed to determine a fit. Both methods fit the FRF fairly well when the correct order is picked.

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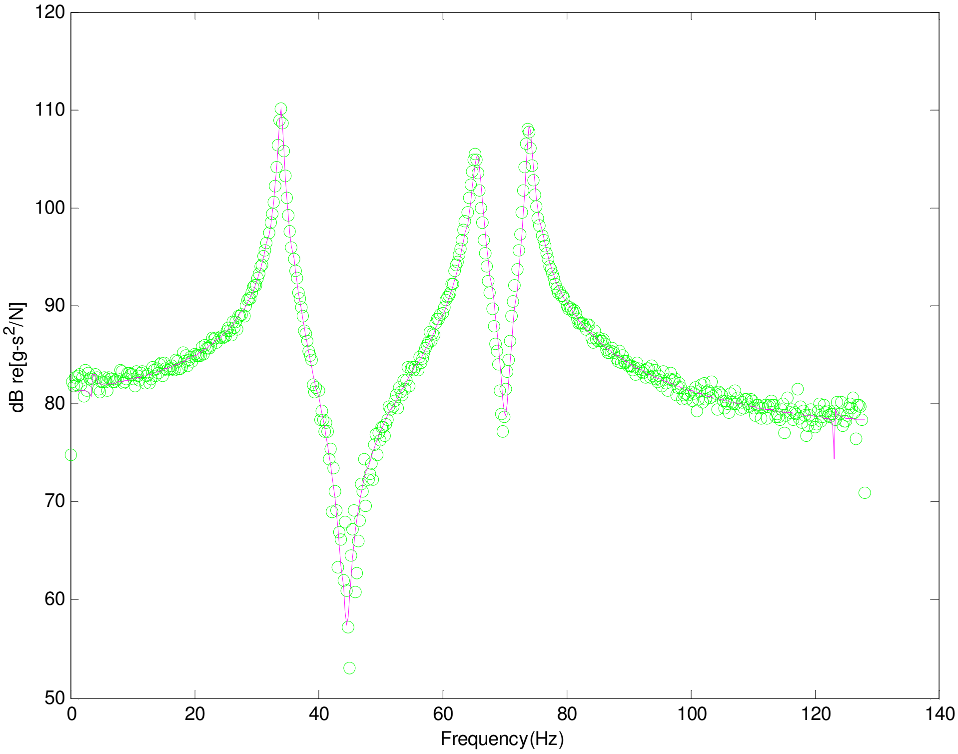

Figure 1: RFP (number of poles = 2 * number of modes)

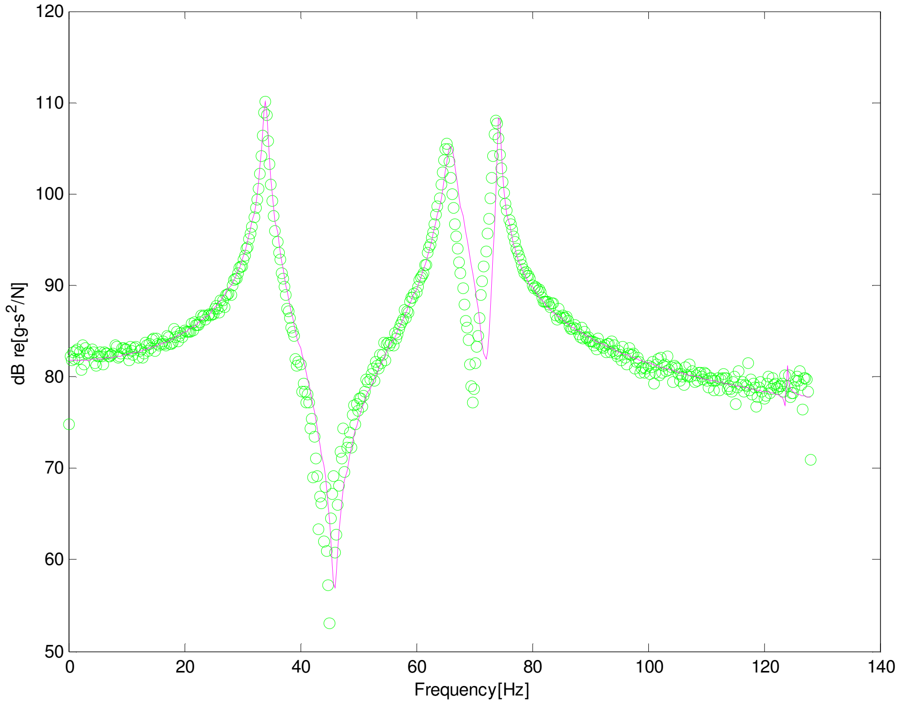

Figure 2: RFOP (No. of poles = 2 * No. of modes)

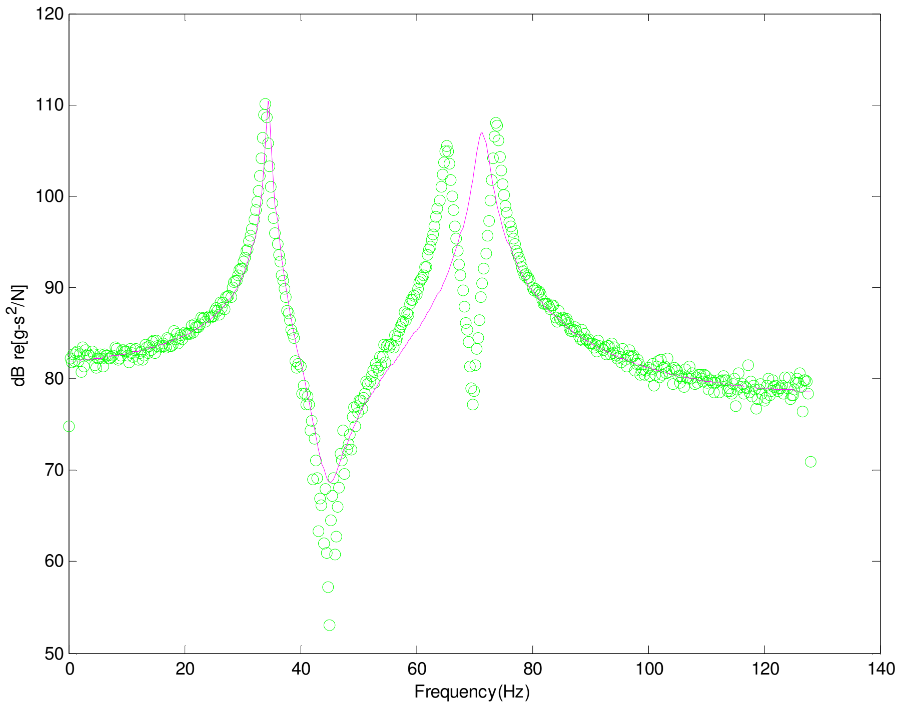

Figure 3: RFP (No. of poles < No. of modes)

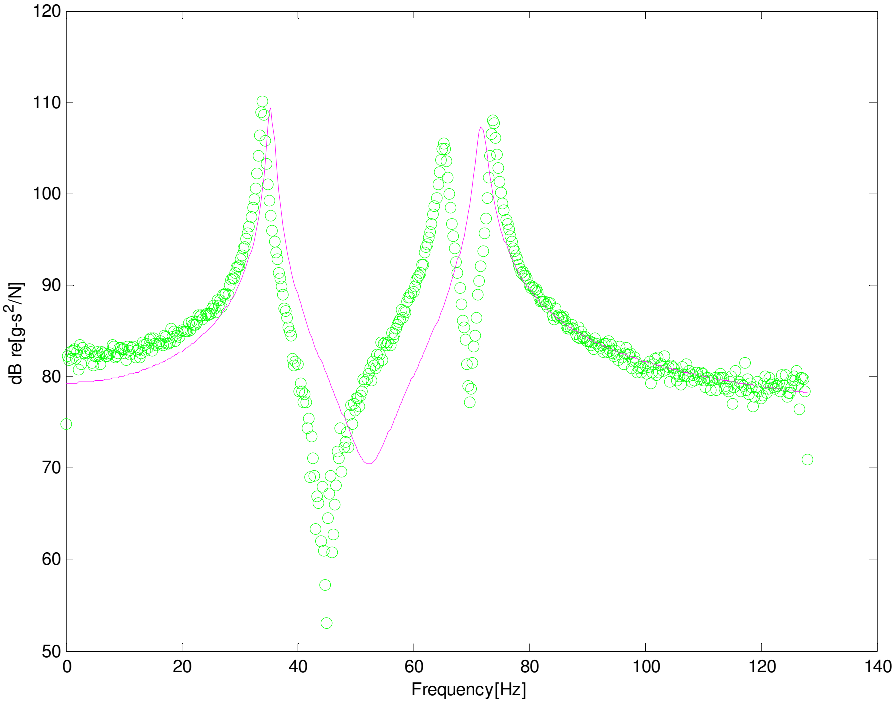

Figure 4: RFOP (No. of poles < No. of modes)

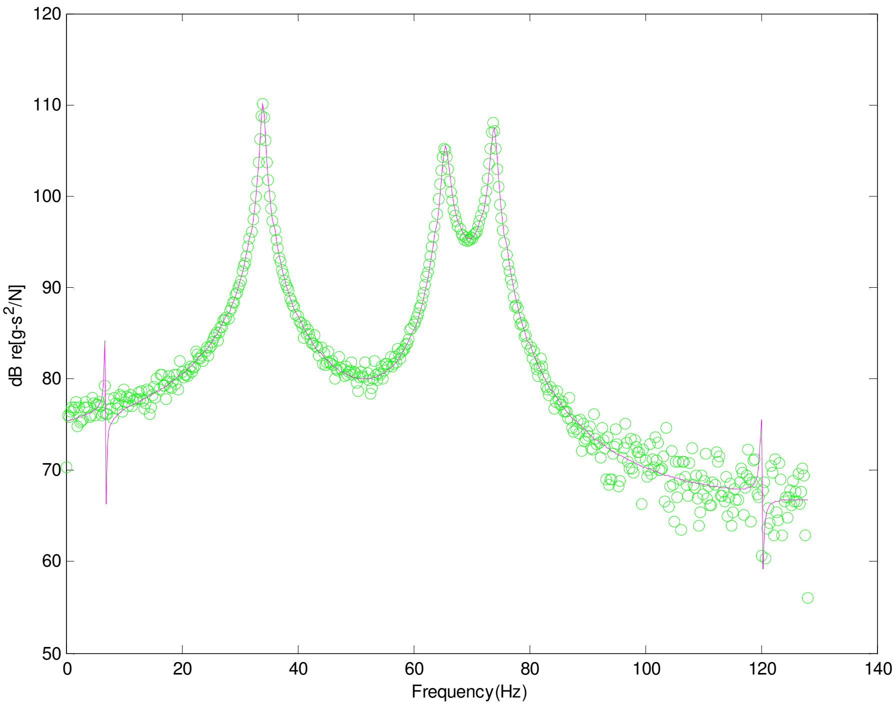

Figure 5: RFP (No. of poles > No. of modes)